A Simplified Mathematical Model of Tidal Effects in a Two-Body System*

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1 Introduction

1.1 Abstract

This paper examines the behaviour of a system composed by two celestial bodies rotating around each other, while they also rotate around their own axes. Such a system evolves dinamically under the influence of the mutual tidal forces. If all the rotations have the same direction, these forces will eventually lead to a synchronization of all the rotations so that the entire system will rotate like a rigid body. The problem is initially treated in general terms calling for convenience Earth and Moon two generic celestial bodies rotating around each other, then the formulas obtained by the model are applied to the case of a simplified Earth-Moon system taking their respective physical and geometrical characteristics into account.

1.2 Simplifying assumptions

1. The two bodies are perfectly spherical and homogeneous and their orbits are circumferences, whose geometrical centres coincide with the centre of mass[1] of the system.
2. The angular rotation speeds of the individual bodies are represented by parallel vectors orthogonal to the plane in which the two bodies rotate.
3. We assume that the system is not perturbed by any outside interactions. As a consequence the total angular momentum[2] of the system will be preserved. Moreover taking the centre of mass of the two bodies as origin O of the (inertial) reference system, also the total momentum is zero.
4. Even though from an energetic point of view it is not possible to consider the system to be totally isolated, because the tidal effects do generate heat and this heat is radiated into space, the momentum and the angular momentum of the system can be reasonably considered constant in the hypothesis that the radiation is emitted in a spherically symmetrical way.
5. The inertial forces generated during the evolution of the system are not considered as usual. In other words the system evolves in the time through a sequence of stationary states of equilibrium between the gravitational forces and the only centrifugal reactions, always mantaining the circular form for both the orbits.

*This paper has been written in LaTeX [1].

[1] Barycentre and centre of mass are very often synonims in scientific books. I prefer to use the last term because the word barycentre contains implicitly a reference to a uniform gravitational field where masses have weights ($\beta\alpha\rho\acute{\upsilon}\varsigma$ = heavy) proportional to the masses themselves.

[2] Such physical quantity is also called total moment of momentum.

2 Symbols

The symbols adopted in this paper are as follows:

Earth:

T = centre of mass of the Earth
m_t = mass of the Earth (kg)
I_t = inertial moment of the Earth around a diametral axis ($kg \cdot m^2$)
ω_t = angular speed of the Earth around its own axis (rad/s)
r_t = radius of the Earth's orbit around O, namely distance OT (m)

Moon:

L = centre of mass of the Moon
m_l = mass of the Moon (kg)
I_l = inertial moment of the Moon around a diametral axis ($kg \cdot m^2$)
ω_l = angular speed of the Moon around its own axis (rad/s)
r_l = radius of the Moon's orbit around O, namely distance OL (m)

Earth-Moon system:

O = centre of mass of the Earth-Moon system
ω_r = orbital angular speed of the Earth and the Moon around O (rad/s)
ω = angular speed in the condition of perfect syncronism (rad/s)

Universe:

G = Newton's gravitational constant = $6.6726 \cdot 10^{-11}\, m^3/(kg \cdot s^2)$

All the rotations are assumed positive if they are anticlockwise for an observer situated over the orbital plane, negative in the opposite case.
In the computations the following parameters are used:

M = total mass of the system (kg):

$$M = m_t + m_l \tag{1}$$

m_r = reduced mass of the system (kg):

$$m_r = \frac{m_t\, m_l}{m_t + m_l} \tag{2}$$

r = distance between the two bodies (m):

$$r = r_t + r_l \tag{3}$$

I = total inertial moment of the two bodies ($kg \cdot m^2$):

$$I = I_t + I_l \tag{4}$$

For convenience we define also the parameter Σ as:

$$\Sigma = G\,M \tag{5}$$

Moreover from the definition of the centre of mass of the two bodies and from the principle of conservation of the momentum, the following relations also hold:

$$r_t = \frac{m_l}{M}\,r \qquad\qquad r_l = \frac{m_t}{M}\,r \tag{6}$$

3 Equations of the gravitational equilibrium

In the hypothesis that the orbits are circular, we can write for each body the equilibrium relation between the gravitational force and the corresponding centrifugal reaction caused by the revolution around O:

$$G\,\frac{m_t\,m_l}{r^2} = m_t\,\omega_r^2\,r_t$$

$$G\,\frac{m_t\,m_l}{r^2} = m_l\,\omega_r^2\,r_l$$

Dividing the first equation for m_t and the second for m_l, then making the addition member by member and keeping into account (1), (3) and (5), we obtain:

$$\frac{\Sigma}{r^2} = \omega_r^2\,r$$

The relation between the Earth-Moon distance and the rotational speed of the system around O is then:

$$r = \left(\frac{\Sigma}{\omega_r^2}\right)^{\frac{1}{3}} \tag{7}$$

As said before, in the hypothesis of an extremely slow evolution of the system, it is possible to disregard the correspondent inertial forces. Therefore r and ω_r will vary with the time, but (7) shall be always verified in any instant of time.

4 Angular momentum

The total angular momentum of the system is the scalar sum of two different contributions[3], the former corresponding to the orbital motion of the two bodies and the latter to the rotation of each body around its own axis.

4.1 Angular momentum due to the orbital motion

For the angular momentum Q' due to the orbital motion around O, we have:

$$Q' = m_t\,\omega_r\,r_t^2 + m_l\,\omega_r\,r_l^2 \tag{8}$$

Substituting (6) and (1) into (8), we have:

$$Q' = \omega_r\,\big(m_t\,\frac{m_l^2}{M^2}\,r^2 + m_l\,\frac{m_t^2}{M^2}\,r^2\big) = \omega_r\,r^2\,\frac{m_t\,m_l\,(m_l + m_t)}{M^2} = \omega_r\,r^2\,\frac{m_t\,m_l}{M}$$

Using the definition of the reduced mass m_r given by (2), we obtain:

$$Q' = m_r\,\omega_r\,r^2 \tag{9}$$

and finally eliminating r by means of (7):

$$Q' = m_r\,\omega_r\,\Big(\frac{\Sigma}{\omega_r^2}\Big)^{\frac{2}{3}} = m_r\,\Big(\frac{\Sigma^2}{\omega_r}\Big)^{\frac{1}{3}}$$

4.2 Angular momentum due the proper rotation of each body

The angular momentum Q'' due to the rotation of each body around its own axis is equal to:

$$Q'' = I_t\,\omega_t + I_l\,\omega_l \tag{10}$$

4.3 Total angular momentum of the system

The total angular momentum Q of the system is therefore the sum of the above calculated contributions:

$$Q = Q' + Q'' = m_r\Big(\frac{\Sigma^2}{\omega_r}\Big)^{\frac{1}{3}} + I_t\,\omega_t + I_l\,\omega_l \tag{11}$$

In the case that all the rotations have not the same sign, the total angular momentum of the system could result very small or also zero, if the two terms compensate each other exactly.

[3] Even if the rotations are vectorial quantities, they will be treated as signed scalars given the simplifying hypothesis that they have and maintain the same direction orthogonal to the orbital plane.

5 Energy of the system

The total energy of the system is the sum of three different contributions: the first is given by the kinetic energy associated to the orbital motion of the two bodies, the second by the kinetic energy due to the proper rotations and finally the third by the potential energy of the gravitational field.

5.1 Energy of the system due to the orbital motion

For the energy of the system E' due to the orbital motion computed with respect to an inertial frame of reference with the origin in O, we have:

$$E' = \frac{1}{2}\, m_t\, \omega_r^2 r_t^2 + \frac{1}{2}\, m_l\, \omega_r^2 r_l^2 \tag{12}$$

Remembering (8) and (9), it is possible to write:

$$E' = \frac{1}{2}\, \omega_r Q' = \frac{1}{2}\, m_r \omega_r^2 r^2 \tag{13}$$

Finally eliminating r^2 with the use of (7), we obtain:

$$E' = \frac{1}{2}\, m_r (\Sigma\, \omega_r)^{\frac{2}{3}} \tag{14}$$

5.2 Energy of the system due to the proper rotations

The kinetic energy E'' due to the rotations of the two bodies around its own axis is given by:

$$E'' = \frac{1}{2}\, I_t\, \omega_t^2 + \frac{1}{2}\, I_l\, \omega_l^2 \tag{15}$$

5.3 Potential energy due to the gravitational field

The gravitational potential energy E_p assumed zero when the two bodies are at infinite distance, is given by:

$$E_p = -G\, \frac{m_t\, m_l}{r} = -G\, \frac{m_r M}{r} = -\frac{m_r \Sigma}{r} \tag{16}$$

Eliminating r with the use of (7), we obtain:

$$E_p = -m_r (\Sigma\, \omega_r)^{\frac{2}{3}} = -2E' \tag{17}$$

5.4 Total energy of the system

The total energy of the system is then given by:

$$E = E' + E'' + E_p = -\frac{1}{2}\, m_r (\Sigma\, \omega_r)^{\frac{2}{3}} + \frac{1}{2}\, I_t\, \omega_t^2 + \frac{1}{2}\, I_l\, \omega_l^2 \tag{18}$$

This energy, whose value E_0 is assumed known at the initial time, can only decrease with the time because the tidal effect warms the bodies and they lose energy by radiation towards the external space.

6 Physico-mathematical model

On the basis of the preceding considerations it is possible to reduce our problem to the solution of a system of total differential equations in three unknown functions, represented by the three angular speeds $\omega_r(t)$, $\omega_t(t)$ e $\omega_l(t)$, having supposed known all the physical data of the two bodies and the values of such functions at the initial time. After the computation of these three functions, the above written formulas make possible the evaluation of all the other correlated quantities.
A first equation of the system is given simply by:

$$Q = Q_0 = \text{constant}$$

where Q_0 is the initial value of the total angular momentum, which will remain constant on the basis of the points 3. and 4. of the paragraph 1.2.
Substituting the expression of Q we obtain:

$$Q_0 = m_r \left(\frac{\Sigma^2}{\omega_r}\right)^{\frac{1}{3}} + I_t\,\omega_t + I_l\,\omega_l \tag{19}$$

The two remaining equations shall represent the energy losses of the system due to the tides. Indicating in general with $E_t(t)$ and with $E_l(t)$ the energies of each of the two bodies, whose variations are caused by the tidal actions, we can write the following equations[4]:

$$\dot{E}_t = -\frac{m_l}{r^n} K_t \cdot |\,\omega_t - \omega_r\,| = -\left(\frac{\omega_r^2}{\Sigma}\right)^{\frac{n}{3}} m_l\, K_t \cdot |\,\omega_t - \omega_r\,| \tag{20}$$

$$\dot{E}_l = -\frac{m_t}{r^n} K_l \cdot |\,\omega_l - \omega_r\,| = -\left(\frac{\omega_r^2}{\Sigma}\right)^{\frac{n}{3}} m_t\, K_l \cdot |\,\omega_l - \omega_r\,| \tag{21}$$

The direct proportionality to the mass of the other body and inverse to a generic n-th power of the distance ($n > 0$), is justified by the fact that the tidal effect on each body is a consequence of the gravitational field generated by the other and certainly decreasing with the increase of the distance. In the most common hypothesis than n is equal to 3 the terms $-m_l/r^3$ and $-m_t/r^3$ would result proportional to the derivative of the gravitational field versus the distance r.
The positive factors K_t e K_l keep into account the radius of the body which undergoes the tidal effect and the mechanical characteristics of the material by which the body is formed (rock, water, solid or melted metallic core, etc.) Finally the absolute differences of the angular speeds keep into consideration the volume of material which undergoes the cyclic tidal deformation per unit of time. As the material constituting the two bodies is not perfectly elastic, the tidal action causes the trasformation in heat of the kinetic-potential energy because of the internal friction, with the consequence that the energy is always decreasing and its derivative shall be negative.
At this point two hypoteses are the most likely:

1. The energies E_t and E_l correspond to the only proper rotation of each body.
2. The energies E_t and E_l include for each body both the part relative to the proper rotation and that associated to the revolution motion around O.

Consequently the first members shall be substituted by the appropriate expressions of the time derivatives of E_t and E_l.
Another possibility could be offered by summing the energy losses of both the bodies, writing the following only equation:

$$\dot{E} = -\left(\frac{\omega_r^2}{\Sigma}\right)^{\frac{n}{3}} \Big(m_t\, K_l \cdot |\,\omega_l - \omega_r\,| + m_l\, K_t \cdot |\,\omega_t - \omega_r\,|\Big) \tag{22}$$

[4]For the derivative with respect to the time we adopt Newton's symbology: $\frac{d\,y}{d\,t} = \dot{y}$, $\frac{d^2\,y}{d\,t^2} = \ddot{y}$.

In this last case there are no doubts about the expression of the first member derivative because E is the total energy given by (18), but a third equation is requested, which specifies how the energy loss is distributed between the two bodies giving in this way a further relation that must be satisfied by the angular speeds.

6.1 Final states of the system

Even if the system of equations above seen in the preceding paragraph can be solved only by numerical methods, nevertheless it is interesting and useful, using the tools of mathematical analysis, to examine the final states towards which the physical system will be evolving and the conditions which make them possible, without solving the equations themselves.
In fact, as the total angular momentum will remain constant with the time on the basis of the equation (19), while in the same time the energy of the system will continue to diminish for the equation (22) (or for the analogous (20) and (21)), until the three angular speeds will coincide, it is possible to foresee that the evolution of the system and the final configurations are only the following three:

1. The two bodies approach each other until they collide, so destroying the original system and generating one only celestial body.
2. The two bodies reach the condition of perfect synchronism when the three angular speeds coincide. This situation, when reached, will be maintained indefinitely, because the tidal dissipative effect disappears completely in such configuration and the system will rotate as an only rigid body with a time-invariant energy.
3. The two bodies, even if losing energy continuously, will go away indefinitely. The increment of the gravitational potential is counterbalanced by a decreasing of the kinetic and rotational energy, while the tidal dissipative effect tends to zero with the increasing of the distance. Nevertheless, as it will demonstrate afterwards, this evolution of the system is incompatible with this model and in particular with the simplifying hypothesis represented by the point 5. of the paragraph 1.2.

We have now to consider the physico-mathematical conditions that determine the evolution of the system towards the first or the second of the above mentioned final configurations on the basis of the values of the physical quantities Q_0 and E_0 that we suppose known at the initial time.

6.1.1 Total angular momentum Q_0 of the system

We now start supposing that the system evolves until it reaches the perfect synchronism situation (case 2 of the preceding paragraph).
In this case, imposing $\omega_r = \omega_t = \omega_l = \omega$, we can rewrite (19) as:

$$Q_0 = m_r \left(\frac{\Sigma^2}{\omega}\right)^{\frac{1}{3}} + I\,\omega \tag{23}$$

In this equation the existence of real zeros for ω is a necessary but not a sufficient condition in order that the system can evolve towards the perfect synchronism, while if all the roots would be pairs of complex coniugate numbers the evolution of the system could bring only to a collision of the bodies, having excluded the third possibility of an indefinite separation of the two bodies.
It wouldn't be difficult to determine analitically all the roots of (23) because it is equivalent to a forth degree polynomial equation, nevertheless it is possible to make some easier considerations in order to verify the existence of real roots. In other words the problem is easily reducible to the

determination of the condition that must be satisfied by Q_0 in order that (23) has or hasn't real solutions.
To this end rewriting (23) and substituting Q_0 with the general symbol of function, we obtain:

$$Q(\omega) \;=\; m_r \Big(\frac{\Sigma^2}{\omega}\Big)^{\frac{1}{3}} + I\,\omega \tag{24}$$

The algebraic function $Q(\omega)$, sum of a $-1/3$ power term and of a linear term, is odd, i.e. it is symmetric as regards the origin; it occupies only quadrants 1 and 3 and has two asymptotes represented by the y axis and by the line $y = I\,\omega$. Its shape is similar to a U, upright in the first quadrant and upside-down in the third.
Therefore the existence of real zeros in (23) corresponds to verify that the line $y = Q_0$ intersects the diagram of (24) at least in one point.
Without losing generality, we can assume Q_0 as positive and consequently we have to consider only the positive part of the function (quadrant 1) with values for ω greater or equal to zero.
The setting to zero of the first derivative and the sign of the second indicates the presence of a relative minimum of the function in corrispondence of the following value $\bar{\omega}_Q$:

$$\bar{\omega}_Q \;=\; \Big(\frac{m_r}{3\,I}\Big)^{\frac{3}{4}} \sqrt{\Sigma} \tag{25}$$

For this value of the angular speed we obtain from (7) the distance $\bar{r}_Q$ between the centres of mass of the bodies, and from (24) the correspondent value Q_{min}:

$$\bar{r}_Q \;=\; \sqrt{3\,\frac{I}{m_r}} \tag{26}$$

$$Q_{min} \;=\; 4\,I\,\bar{\omega}_Q \;=\; 4\,\Big(\frac{m_r}{3}\Big)^{\frac{3}{4}} I^{\frac{1}{4}} \sqrt{\Sigma} \tag{27}$$

It is noticeable that Q_{min} depends only on the global physico-geometrical characteristics of the two bodies, i.e. on the reduced mass, the total inertial moment and Σ, that is the product of the total mass and the universal gravitational constant. Moreover in this particular situation the angular momentum is due for 3/4 to the orbital motion and for 1/4 to their proper rotations of the two bodies.
As there is one sole minimum in the positive field of ω, a perfect synchronism configuration of the system will be possible only if $Q_0 \;\geq\; Q_{min}$.
As a consequence if $Q_0 = Q_{min}$ equation (24) will have one double real root, while for values of Q_0 greater than Q_{min} two distinct real roots will exist, the first at left and the second at right of $\bar{\omega}_Q$, i.e.:

$$\omega_1 \;<\; \bar{\omega}_Q \;<\; \omega_2$$

About the distance r between the centres of mass of the two bodies, it can be easily calculated by (7) for each values of ω.
Therefore on the basis of the only value of Q_0, the following conclusions can be done:

1. $Q_0 \;>\; Q_{min}$: two possible conditions of perfect synchronism exist, one of which, as it will be demonstrated further, is stable because it corresponds to a minimum of the total energy, while the other is instable, because such condition is not verified.

2. $Q_0 \;=\; Q_{min}$: one only condition of perfect synchronism exists, whose stability or instability cannot be easily determined.

3. $Q_0 \;<\; Q_{min}$: no perfect synchronism configuration is possible and the system will evolve towards a collision of the two bodies.

6.1.2 Total initial energy E_0 of the system

The total initial energy E_0 of the system permits us of introducing further limitations to the conditions $Q_0 = Q_{min}$ or $Q_0 > Q_{min}$ which established the existence of one or two perfect synchronism configurations.
We can write also in this case the expression of the total energy in such particular situations, rewriting (18) with $\omega_r = \omega_t = \omega_l = \omega$:

$$E(\omega) = -\frac{1}{2}\, m_r\, (\Sigma\, \omega)^{\frac{2}{3}} + \frac{1}{2}\, I\, \omega^2 \tag{28}$$

This function is even, i.e. symmetric in respect to the y axis and has a sharp-cornered point in the origin where two derivatives exist (left derivative $= +\infty$, right derivative $= -\infty$). Also in this case, without losing generality, we can limit us to consider only the positive values of ω.
The setting to zero of the first derivative and the sign of the second indicates the presence of a relative minimum of the function in corrispondence of the following value $\bar{\omega}_E$:

$$\bar{\omega}_E = \left(\frac{m_r}{3\, I}\right)^{\frac{3}{4}} \sqrt{\Sigma} \tag{29}$$

It is noticeable that this value of ω is exactly the same found in correspondence of the minimum of the angular momentum. The value of E_{min} is then given by:

$$E_{min} = -\frac{3}{2}\, I\, \bar{\omega}_E^2 + \frac{1}{2}\, I\, \bar{\omega}_E^2 = -I\, \bar{\omega}_E^2 = -\left(\frac{m_r}{3}\right)^{\frac{3}{2}} \frac{\Sigma}{\sqrt{I}} \tag{30}$$

Also in this case the energy due to the gravitational potential summed with the kinetic translational energy is in modulus three times the kinetic energy due to the proper rotations of the two bodies. This observation is less significative of the analogous one regarding the angular momentum, as the potential energy is defined introducing an arbitrary constant. Finally it is noticeable that also E_{min}, as Q_{min}, depends only on the global physico-geometrical characteristics of the two bodies.
For what has been said in the preceding paragraph, in the hypothesis that $Q_0 > Q_{min}$, is satisfied, the real solutions of algebraic equation (23) represent the two angular speeds ω_1 and ω_2 ($\omega_1 < \omega_2$), correspondent to the two configurations of perfect synchronism.
Using these values in (28) we can find the energies E_1 and E_2 where we suppose $E_2 > E_1$ [5].
Considering the value of E_0 it is possible to say that:

1. $E_0 \geq E_2$: both the perfect synchronism configurations are possible.
2. $E_2 > E_0 \geq E_1$: only one perfect synchronism configuration is reachable by the system, that corresponds to the angular speed ω_1.
3. $E_0 < E_1$: no perfect synchronism configuration is possible and the system will surely evolve towards a collision of the two bodies.

In the particular case $Q_0 = Q_{min}$ equation (23) gives the double root $\omega_1 = \omega_2 = \bar{\omega}_Q$ and consequently $E_1 = E_2 = E_{min}$; the existence of one perfect synchronism configurations is then conditioned by the relation $E_0 \geq E_{min}$.
As final comment to this paragraph, we observe that differently from the angular momentum which remains constant with the time, the total energy of the system is in general continuously decreasing. As a consequence, the above mentioned conclusions that refer to the initial energy E_0 are only necessary but non sufficient conditions for reaching the perfect synchronism configuration. In the successive paragraph the problem is examined in more general terms.

[5] This inequality holds surely for values of Q_0 larger of a certain limit, while remains indetermined for smaller values.

6.2 Total energy of the system: study of the function

The total energy of the system defined by (18) would seem to depend on the three angular speeds contained in the equation. In reality the independent variables are only two because we can use (19) in order to make explicit the variable ω_t :

$$\omega_t \;=\; \frac{1}{I_t}\Big(Q_0 - m_r\Big(\frac{\Sigma^2}{\omega_r}\Big)^{\frac{1}{3}} - I_l\,\omega_l\Big) \tag{31}$$

Therefore we can rewrite the relation (18) assuming it as function of the only variables ω_r and ω_l, having ideally done the substitution of ω_t by means of (31):

$$E(\omega_r,\omega_l) \;=\; -\frac{1}{2}\,m_r(\Sigma\,\omega_r)^{\frac{2}{3}} + \frac{1}{2}\,I_t\,\omega_t^2(\omega_r,\omega_l) + \frac{1}{2}\,I_l\,\omega_l^2 \tag{32}$$

Now we want to analyze this function in order to calculate the characteristic point (or points) of the correspondent 3-d surface. Making equal to zero the two first partial derivatives we can write:

$$\frac{\partial\,E}{\partial\,\omega_r} \;=\; -\frac{1}{3}\,m_r\Big(\frac{\Sigma^2}{\omega_r}\Big)^{\frac{1}{3}} + I_t\,\omega_t\,\frac{\partial\,\omega_t}{\partial\,\omega_r} \;=\; 0 \tag{33}$$

$$\frac{\partial\,E}{\partial\,\omega_l} \;=\; I_t\,\omega_t\,\frac{\partial\,\omega_t}{\partial\,\omega_l} + I_l\,\omega_l \;=\; 0 \tag{34}$$

Substituting the ω_t variable and its derivatives obtained from (31), the two preceding relations become:

$$\frac{1}{3}\,m_r\Big(\frac{\Sigma}{\omega_r^2}\Big)^{\frac{2}{3}}\,(\omega_t\,-\,\omega_r) \;=\; 0 \tag{35}$$

$$I_l\,(\omega_l\,-\,\omega_t) \;=\; 0 \tag{36}$$

It is immediate to see that the first equation is satisfied if $\omega_t = \omega_r$ and the second if $\omega_l = \omega_t$, i.e. if the three angular speeds are all equal, and this condition, together with (31), determines the only admissible solutions[6]. Such solutions coincide with the roots of the equation (23) and then there are 0, 1 or 2 real solutions depending on the value of Q_0.

The calculation of the second partial derivatives gives the following relations:

$$\frac{\partial^2 E}{\partial\,\omega_r^2} = \frac{1}{9}\,m_r\Big(\frac{\Sigma}{\omega_r^2}\Big)^{\frac{2}{3}}\left[1 - 4\,\frac{\omega_t}{\omega_r} + \frac{m_r}{I_t}\,\Big(\frac{\Sigma}{\omega_r^2}\Big)^{\frac{2}{3}}\right] \tag{37}$$

$$\frac{\partial^2 E}{\partial\,\omega_l^2} = \frac{I_l^2}{I_t} + I_l \tag{38}$$

$$\frac{\partial^2 E}{\partial\omega_r\,\partial\omega_l^2} = -\frac{1}{3}\,m_r\,\frac{I_l}{I_t}\,\Big(\frac{\Sigma}{\omega_r^2}\Big)^{\frac{2}{3}} \tag{39}$$

The zeros of the first partial derivatives correspond effectively to a relative minimum of the function if the following inequalities are satisfied:

$$\frac{\partial^2 E}{\partial\,\omega_r^2}\,\frac{\partial^2 E}{\partial\,\omega_l^2} - \Big(\frac{\partial^2 E}{\partial\omega_r\,\partial\omega_l}\Big)^2 \;>\; 0 \tag{40}$$

[6]For evident reasons the banal solution $\omega_r = \infty$ and $\omega_l = \omega_t = Q_0/I$ correspondent to a null distance between the two bodies is not taken into account.

$$\frac{\partial^2 E}{\partial\,\omega_l^2} \; > \; 0 \tag{41}$$

The first inequality[7], if satisfied, assure us that a relative minimum or maximum exists, while the second indicates that such point is a minimum.
But as from (38) it is obvious that the second inequality is always and in any case satisfied, then it is possible to exclude the existence of a relative maximum in the previously found point, while the condition for the existence of a relative minimum, after the substitutions in (40), becomes:

$$\frac{1}{9}\,m_r\,\frac{I_l}{I_t}\left(\frac{\Sigma}{\omega_r^2}\right)^{\frac{2}{3}}\left[I\left(1-4\,\frac{\omega_t}{\omega_r}\right)+m_r\left(\frac{\Sigma}{\omega_r^2}\right)^{\frac{2}{3}}\right] \; > \; 0 \tag{42}$$

Having excluded the case $\omega_r = \infty$, after division for the certainly positive common factor and equalization of the angular speeds ($\omega_r = \omega_t = \omega$), we obtain:

$$-3\,I + m_r\left(\frac{\Sigma}{\omega^2}\right)^{\frac{2}{3}} \; > \; 0 \tag{43}$$

Rearranging this inequality and comparing the result with (25) we can immediately write:

$$\omega \; < \; \left(\frac{m_r}{3\,I}\right)^{\frac{3}{4}}\sqrt{\Sigma} \; = \; \bar{\omega}_Q \tag{44}$$

In this way it is justified the statement that if $Q_0 \; > \; Q_{min}$, between the two possible perfect synchronism configurations, one only is stable because it corresponds to a relative minimum of the total energy, while the other is instable, because it is associated to a saddle point of such function.
In conclusion the stable solution corresponds to an angular speed lesser than $\bar{\omega}_Q$ and therefore to a radius greater than $\bar{r}_Q$. Using (7) and (26) we can write:

$$r \; > \; \sqrt{3\,\frac{I}{m_r}} \; = \; \bar{r}_Q \tag{45}$$

It is possible to say that if the perfect synchronism condition corresponds to a distance between the centres of mass of the two bodies that satisfied (45), then in such point the energy has a relative minimum and the configuration of the system is stable.
On the contrary if $r < \sqrt{3\,I/m_r}$ the function presents a saddle point and consequently a state of instable equilibrium, while nothing can be said if an equal sign holds.
Syntetically we can summarize the preceding considerations showing the conditions that establish if an initially not synchronous two-body system can evolve with the time towards a perfect synchronism condition:

- The total angular momentum Q_0 of the system, initially known and invariant with the time in the approximation of this model, must be greater than a minimum value satisfying the following condition:
$$Q_0 \; > \; 4\left(\frac{m_r}{3}\right)^{\frac{3}{4}} I^{\frac{1}{4}}\,\sqrt{\Sigma}$$
- In correspondence to the value Q_0 the equation (23) will have two real distinct roots ω_1 and ω_2 with $\omega_1 < \omega_2$, of which only the lesser corresponds to a relative minimum of the total energy and therefore to a stable state of perfect synchronism.

[7]The left member of this inequality is called in the literature the Hessian of the function: if the Hessian is positive in a point where the first partial derivatives of a two-variable function are zero, then the function has a relative minimum or maximun in that point; on the contrary, if the Hessian is negative, the function has no minimum or maximum. Finally if the Hessian is null, any conclusion is postponed to the evaluation of the higher order partial derivatives.

- The total initial energy E_0 must be necessarily greater than the energy correspondent to the angular speed ω_1 in the perfect synchronism configuration satisfying the following condition:

$$E_0 > -\frac{1}{2}\, m_r\, (\Sigma\, \omega_1)^{\frac{2}{3}} + \frac{1}{2}\, I\, \omega_1^2$$

  The last relation is a necessary but not sufficient condition because the same inequality shall be satisfied in general by the total energy $E(t)$, which is a decreasing function of the time, in any instant of the evolution, from the beginning ($t = 0$) until the reaching of the perfect synchronism condition.

  Therefore the final situation will depend also on the particular form of the equations (20) and (21), examined but not exhaustively analyzed in a preceding chapter.

7 The Earth-Moon system

In the particular case of the Earth-Moon system the equations that define the energy losses are notably simplified, because the proper rotation of the Moon is already synchronous with the motion of revolution of the two bodies around the centre of mass O of the system, with the consequence that the Moon shows always the same face to the Earth[8].
In the very likely hypothesis that such synchronism is maintained with the time, we can rewrite the preceding equations assuming $\omega_l = \omega_r$:

1. Conservation of the angular momentum ($kg \cdot m^2 \cdot s$):

$$Q_0 \;=\; m_r \Big(\frac{\Sigma^2}{\omega_r}\Big)^{\frac{1}{3}} + I_t\,\omega_t + I_l\,\omega_r \tag{46}$$

2. Total energy of the system (J):

$$E \;=\; -\frac{1}{2}\, m_r (\Sigma\,\omega_r)^{\frac{2}{3}} + \frac{1}{2}\, I_t\,\omega_t^2 + \frac{1}{2}\, I_l\,\omega_r^2 \tag{47}$$

3. Energy loss for tidal effect (W):

$$\dot{E} \;=\; -\Big(\frac{\omega_r^2}{\Sigma}\Big)^{\frac{n}{3}} m_l\, K_t\; \cdot |\,\omega_t - \omega_r\,| \tag{48}$$

Equation (48) derives directly from (22) where the dissipative term due to the tidal action of the Earth on the Moon disappears because of the spin-orbit synchronism of the Moon. In fact in this situation the gravitational force of the Earth determines a permanent ovalization of the Moon with the major axis in the direction of the Earth, but without any energy loss, apart the initial cost of the deformation work.
Substituting in (48) the expression obtained from the time derivative of (47), we obtain:

$$\Big(-\frac{1}{3}\, m_r \Big(\frac{\Sigma^2}{\omega_r}\Big)^{\frac{1}{3}} + I_l\,\omega_r\Big)\,\dot{\omega}_r + I_t\,\omega_t\,\dot{\omega}_t \;=\; -\Big(\frac{\omega_r^2}{\Sigma}\Big)^{\frac{n}{3}} m_l\, K_t\; \cdot |\,\omega_t - \omega_r\,| \tag{49}$$

7.1 Analytical solution of the equation of the Earth-Moon system

The system of equations in the two unknowns ω_r and ω_t represented by (46) and (49) is perfectly defined with the knowledge of the initial conditions and all the physical and kinematic parameters of the system, and is easily solvable by analytical methods.
To this aim we must notice that the modulus sign present in (49), can be deleted *sic et simpliciter* substituting $|\,\omega_t - \omega_r\,|$ with $(\omega_t - \omega_r)$.
In fact such difference is at present positive because the Earth rotates around its axis more rapidly than the two-body system around its centre of mass O, and the two angular speeds have both the same positive sign in the chosen frame of reference.
Moreover it is not physically possible that such expression can change its sign in a discontinue way, because the acting forces are finite and operate on masses with a large translational and rotational inertial moments.
When with the time the difference will become zero, at the same moment also the tidal friction energy loss will disappear and the system will assume the perfect synchronism stable state, rotating as a sole rigid body without any further energy loss.

[8] In reality, due to the eccentricity of its orbit and to other minor effects, the Moon shows us about 59% of its surface.

Taking the derivative of (46) we obtain:

$$I_t\,\dot\omega_t \;=\; \Big(\frac{1}{3}\,m_r\Big(\frac{\Sigma}{\omega_r^2}\Big)^{\frac{2}{3}} - I_l\Big)\,\dot\omega_r \tag{50}$$

Substituting this expression in (49) and removing the modulus sign, we have:

$$\Big(-\frac{1}{3}\,m_r\Big(\frac{\Sigma^2}{\omega_r}\Big)^{\frac{1}{3}} + I_l\,\omega_r\Big)\,\dot\omega_r + \omega_t\,\Big(\frac{1}{3}\,m_r\Big(\frac{\Sigma}{\omega_r^2}\Big)^{\frac{2}{3}} - I_l\Big)\,\dot\omega_r \;=\; -\Big(\frac{\omega_r^2}{\Sigma}\Big)^{\frac{n}{3}}\,m_l\,K_t\,(\,\omega_t-\omega_r\,) \tag{51}$$

Finally after some simplifications it is possible to divide both the member by the factor $(\omega_t-\omega_r)$ obtaining an equation in the only unknown ω_r:

$$\dot\omega_r\,\Big(\frac{1}{3}\,m_r\Big(\frac{\Sigma}{\omega_r^2}\Big)^{\frac{2}{3}} - I_l\Big) \;=\; -\Big(\frac{\omega_r^2}{\Sigma}\Big)^{\frac{n}{3}}\,m_l\,K_t \tag{52}$$

The division by $(\omega_t-\omega_r)$ is mathematically correct as long as such factor differs from zero, considering that the annulment can happen only and exactly in the instant of time when the computation should terminate in all those cases in which the evolution of the system reaches the condition of perfect synchronism.
Equation (52) is a differential equation in one unknown associated with an algebraic equation given by (46), which defines the expression of ω_t:

$$\omega_t \;=\; \frac{1}{I_t}\,\Big(Q_0 - m_r\,\Big(\frac{\Sigma^2}{\omega_r}\Big)^{\frac{1}{3}} - I_l\,\omega_r\Big)$$

Rewriting the derivative $\dot\omega_r$ in (52) as ratio of differentials it is evident that we have to solve the following differential separable variables equation:

$$\Big(I_l - \frac{1}{3}\,m_r\,\Big(\frac{\Sigma}{\omega_r^2}\Big)^{\frac{2}{3}}\Big)\Big/\Big(\Big(\frac{\omega_r^2}{\Sigma}\Big)^{\frac{n}{3}}\,m_l\,K_t\Big)\,d\omega_r \;=\; dt$$

Putting in evidence the powers of ω_r the equation to be solved appears in an extremely simple form:

$$\frac{I_l\,\Sigma^{\frac{n}{3}}}{m_l\,K_t}\;\omega_r^{-\frac{2n}{3}}\,d\omega_r - \frac{m_r\,\Sigma^{\frac{2+n}{3}}}{3\,m_l\,K_t}\;\omega_r^{-\frac{2n+4}{3}}\,d\omega_r \;=\; dt$$

Introducing the constant A the solution is given by the function:

$$-\frac{3}{2\,n-3}\;\frac{I_l\,\Sigma^{\frac{n}{3}}}{m_l\,K_t}\;\omega_r^{-\frac{2n-3}{3}} + \frac{1}{2\,n+1}\;\frac{m_r\,\Sigma^{\frac{2+n}{3}}}{m_l\,K_t}\;\omega_r^{-\frac{2n+1}{3}} + A \;=\; t \tag{53}$$

From now forward we will use this relation assuming $n=3$, which is the most common adopted value and corresponds to the hypothesis that the energy loss for tidal effect is proportional to the gradient of the gravitational field produced by the other body.
In this case (53) becomes:

$$t \;=\; A - \frac{I_l\,\Sigma}{m_l\,K_t}\;\omega_r^{-1} + \frac{1}{7}\;\frac{m_r\,\Sigma^{\frac{5}{3}}}{m_l\,K_t}\;\omega_r^{-\frac{7}{3}} \tag{54}$$

Relation (54) represents in explicit form the function $t = t(\omega_r)$ which associates the time, considered as dependent variable, to the angolar speed of revolution seen as independent variable. From a physical point of view it would be better to have the inverse function $\omega_r = \omega_r(t)$, but unfortunately this last function is not analitically representable because it would request the solution of a seventh degree algebraic equation.
The value of A must be determined by means of the initial condition; assuming $t = 0$ for the present time, we have $\omega_r(0) = \omega_{r,o} = 2.6617 \cdot 10^{-6}$ rad/s, which is the present angular speed of rotation and revolution of the Moon.
Therefore the constant A has the following expression:

$$A = \frac{\Sigma}{m_l\, K_t\, \omega_{r,o}} \left(I_l - \frac{1}{7}\, m_r \left(\frac{\Sigma}{\omega_{r,o}^2} \right)^{\frac{2}{3}} \right) \qquad (55)$$

On the preceding developments it is possible to make some comments:

- As we said before the equation (52) has been obtained dividing both the members for $(\omega_t - \omega_r)$ in the hypothesis that such term is always positive during the system evolution and therefore the modulus sign would have been ininfluential in this case. In order to keep into account a possible change in the sign, we could have used the term $\sqrt{(\omega_t - \omega_r)^2}$ but this would have made the equation analitically unsolvable. However this choice is not necessary, because, as it has been actually done, it is easier to divide the integration interval of the differential equation in two or more parts where $(\omega_t - \omega_r)$ doesn't change the sign.

  In the case $|\,\omega_t - \omega_r\,| = -(\,\omega_t - \omega_r\,)$ the solution (54) remains equally valid with only a change of the sign in one of the two members.

- The given solution is correct also for values of ω_r greater of the present value, i.e. for negative values of the time, if we suppose that in the past the Moon had already reached the present spin-orbit synchronism.

- From (54) and (55) it is evident that the time t is inversely proportional to the constant K_t. This parameter, whose determination could seem somewhat difficult, is in reality easily evaluable if we could know the present total power dissipated for tidal effect by the system, in this case by the only Earth. In its turn such power is clearly correlated with the variation of the length of the sideral terrestrial day, whose value is available in literature even if with a certain margin of error. In conclusion the parameter K_t, though not directly measurable, is easily obtainable from this last quantity, as we will do in the next paragraph.

- The formula (22) used for evaluating the energy loss could be substituted by a sum of similar terms, each with a different value of n, as normally happens in a power series development. Also in this case an analytical solution should be obtainable as the problem can be reduced to the integration of a rational function.

7.2 Final state of the Earth-Moon system

In what follows we have reported the results of calculations obtained by the application of the theory and the formulas previously written.
It is important to stress that the results remain *always and in any case purely indicatory*, even if they use precise astronomical and scientific data, because of the numerous semplifying hypotheses of the model. In particular the presence of a third celestial body, the Sun, it is cause of not negligible tidal effects that interfere with the Earth-Moon system.
Anyhow it is interesting to notice that the conclusions of this very simple model are coherent with those reported in the literature, even if obtained with more sophisticated models and calculations.

The present physical and orbital data of the Earth-Moon system are as follows:

Earth:

m_t = mass: $5.9742 \cdot 10^{24}\ kg$
I_t = inertial moment[9]: $8.0285 \cdot 10^{37}\ kg \cdot m^2$
$\omega_{t,o}$ = angular speed around its proper axis: $+7.29212 \cdot 10^{-5}\ rad/s$
$T_{t,o}$ = rotation period around its proper axis: $8.6164 \cdot 10^4 s$
$r_{t,o}$ = radius of the terrestrial orbit around O: $4.675053 \cdot 10^6\ m$

Moon:

m_l = mass: $7.3483 \cdot 10^{22}\ kg$
I_l = inertial moment: $8.6821 \cdot 10^{34}\ kg \cdot m^2$
$\omega_{l,o}$ = angular speed around its proper axis: $+2.6617 \cdot 10^{-6}\ rad/s$
$T_{l,o}$ = rotation period around its proper axis: $2.360592 \cdot 10^6 s$
$r_{l,o}$ = radius of the lunar orbit around O: $3.800839 \cdot 10^8\ m$

Earth-Moon system:

M = toal mass: $6.04768 \cdot 10^{24}\ kg$
m_r = reduced mass: $7.25901 \cdot 10^{22}\ kg$
I = total inertial moment: $8.03719 \cdot 10^{37}\ kg \cdot m^2$
$\omega_{r,o}$ = angular speed of revolution of the system around O: $+2.6617 \cdot 10^{-6}\ rad/s$
$T_{r,o}$ = period of revolution of Moon and Earth around O: $2.360592 \cdot 10^6 s$
r_o = mean distance between the centres of Moon and Earth: $3.847590 \cdot 10^8\ m$
$\Sigma\ =\ GM$: $4.03538 \cdot 10^{14}\ m^3/s^2$

Universe:

G = Newton's gravitational constant: $6.6726 \cdot 10^{-11}\ m^3/(kg \cdot s^2)$

Using these data in (46) and (47) of the preceding paragraphs, we obtain the present values of the total angular momentum and of the total energy:

$$Q_0 \;=\; m_r \Big(\frac{\Sigma^2}{\omega_{r,o}}\Big)^{\frac{1}{3}} + I_t\,\omega_{t,o} + I_l\,\omega_{r,o} \;=\; 3.4458 \cdot 10^{34}\ kg \cdot m^2/s$$

$$E_0 \;=\; -\frac{1}{2}\,m_r(\Sigma\,\omega_{r,o})^{\frac{2}{3}} + \frac{1}{2}\,I_t\,\omega_{t,o}^2 + \frac{1}{2}\,I_l\,\omega_{r,o}^2 \;=\; 1.7539 \cdot 10^{29}\ J$$

[9]This value and the analogous for the Moon are taken from the literature and keep into account the non-homogeneous distribution of the density of the two bodies.

With only the physical parameters of the Earth-Moon system, it is possible to compute the characteristic values correspondent to the relative minimum in the perfect synchronism condition:

$$\bar{r}_Q = \sqrt{3\,\frac{I}{m_r}} = 5.7633 \cdot 10^7 \; m = 57\,633 \; km$$

$$\bar{\omega}_Q \equiv \bar{\omega}_E = \left(\frac{m_r}{3\,I}\right)^{\frac{3}{4}} \sqrt{\Sigma} = 4,5913 \cdot 10^{-5} \; rad/s$$

$$\bar{T}_Q = \frac{2\,\pi}{\bar{\omega}_Q} = 1.3685 \cdot 10^5 \; s = 1.58 \;\; days$$

$$Q_{min} = 4\,I\,\bar{\omega}_Q = 1.4760 \cdot 10^{34} \; kg \cdot m^2/s$$

$$E_{min} = -I\,\bar{\omega}_E^2 = -1.6942 \cdot 10^{29} \; J$$

In order to establish the situation towards which the Earth-Moon system will evolve, it is necessary to solve the equation (23) sustituting the known physical quantities with the above mentioned values, so obtaining the following equation:

$$3.44579 \cdot 10^{34} = 3.9640 \cdot 10^{32} \cdot \omega^{-1/3} + 8.03719 \cdot 10^{37} \cdot \omega$$

This equation is equivalent to an algebraic forth degree equation, and could be solved analytically; nevertheless it is more simple and convenient to find its real roots by usual numerical methods. As $Q_0 > Q_{min}$ the equation will have two real roots:

$$\omega_1 = 1.5389 \cdot 10^{-6} \; rad/s \qquad\qquad \omega_2 = 3.5936 \cdot 10^{-4} \; rad/s$$

The correspondent quantities, period of rotation/revolution, distance between the two bodies and total system energy, are as follows:

$$T_1 = 47.25 \;\; days \qquad r_1 = 554\,388 \; km \qquad E_1 = -\,2.6324 \cdot 10^{28} \; J$$

$$T_2 = 0.20 \;\; days \qquad r_2 = 16\,620 \; km \qquad E_2 = +\,4.1877 \cdot 10^{30} \; J$$

As E_2 is greater than E_0, the second solution is physically impossible; only the first, for which we have $E_0 > E_1$, represents a final allowable situation, which therefore can be reached by the system. From the experimental knowledge that the sideral day increases of $14.6 \cdot 10^{-4}$ s per century, it is possible to calibrate the value of the parameter K_t, and then compute the power P presently produced by the solid and liquid terrestrial tides.

Using an electronic worksheet we obtain the following values:

$$P = 2.2086 \cdot 10^{12} \; W = 2.2086 \; TW$$

$$K_t = 2.4367 \cdot 10^{19} \; m^5/s^2$$

$$A = -1.2998 \cdot 10^{17} \; s$$

On the basis of this model, the total power of the earthly tides would be equivalent to 2209 power plants each of them with a power of 1000 (thermal) MW.

7.2.1 Future evolution of the Earth-Moon system

The following table is built using the above mentioned formulas:

Table 1: Future evolution of the Earth-Moon system

Orbital Angular Speed ω_r (rad/s)	Rotational Earth Speed ω_t (rad/s)	Elapsed Time t $(years)$	Earth-Moon Distance r (km)	Total Energy E (J)	Energy Difference $E - E_{min}$ (J)
$2.6617 \cdot 10^{-6}$	$7.2921 \cdot 10^{-5}$	0	384 759	$+1.7539 \cdot 10^{29}$	$2.0172 \cdot 10^{29}$
$2.6000 \cdot 10^{-6}$	$7.0125 \cdot 10^{-5}$	$2.3169 \cdot 10^{8}$	390 822	$+1.5993 \cdot 10^{29}$	$1.8625 \cdot 10^{29}$
$2.4000 \cdot 10^{-6}$	$6.0416 \cdot 10^{-5}$	$1.1251 \cdot 10^{9}$	412 244	$+1.1100 \cdot 10^{29}$	$1.3732 \cdot 10^{29}$
$2.2000 \cdot 10^{-6}$	$4.9564 \cdot 10^{-5}$	$2.3055 \cdot 10^{9}$	436 864	$+6.5087 \cdot 10^{28}$	$9.1411 \cdot 10^{28}$
$2.0000 \cdot 10^{-6}$	$3.7310 \cdot 10^{-5}$	$3.9056 \cdot 10^{9}$	465 523	$+2.4417 \cdot 10^{28}$	$5.0741 \cdot 10^{28}$
$1.8000 \cdot 10^{-6}$	$2.3302 \cdot 10^{-5}$	$6.1420 \cdot 10^{9}$	499 398	$-7.5307 \cdot 10^{27}$	$1.8793 \cdot 10^{28}$
$1.6000 \cdot 10^{-6}$	$7.0500 \cdot 10^{-6}$	$9.3875 \cdot 10^{9}$	540 192	$-2.5118 \cdot 10^{28}$	$1.2058 \cdot 10^{27}$
$1.5389 \cdot 10^{-6}$	$1.5389 \cdot 10^{-6}$	$1.0671 \cdot 10^{10}$	554 388	$-2.6324 \cdot 10^{28}$	0
$1.5000 \cdot 10^{-6}$	$-2.1298 \cdot 10^{-6}$	$1.1583 \cdot 10^{10}$	563 941	$-2.5789 \cdot 10^{28}$	$5.3453 \cdot 10^{26}$

The last line of the table, which corresponds to a configuration beyond the perfect synchronism point, is not physically and mathematically correct, because at the preceding line, where such synchronism is reached, the difference $\omega_t - \omega_r$ is zero and this determines the end of any tidal effect on the two-body system; moreover in such point (52) is no more valid due to the division by zero.
The line has been equally left in tha table in order to show the fact that in correspondence to the angular speed ω_1 there is actually a minimum for the system energy and therefore a stable equilibrium condition.
In conclusion, in the symplifying hypotheses of this model, the perfect synchronism condition with the angular speed ω_1 is a stable equilibrium configuration that would be reached in about ten and half billion years.

7.2.2 Past evolution of the Earth-Moon system

Only in the hypothesis that the spin-orbit synchronism had been already reached, it is possible to use the same resolutive formula in order to reconstruct the past Earth-Moon system evolution.
So, even if such hypothesis must be considered almost certainly wrong, because it is unlikely that the Earth-Moon system formed with such synchronism already present, nevertheless it is probable the the synchronization process of the Moon happened since the beginning in an astronomically short interval of time considering the strong tidal action of the Earth on the lunar rocks. As a consequence the following table can be assumed as valid at least in a not very far past.

Table 2: Past evolution of the Earth-Moon system

Orbital Angular Speed ω_r (rad/s)	Rotational Earth Speed ω_t (rad/s)	Elapsed Time t $(years)$	Earth-Moon Distance r (km)	Total Energy E (J)	Energy Difference $E_{max} - E$ (J)
$2.6617 \cdot 10^{-6}$	$7.2921 \cdot 10^{-5}$	0	384 759	$+1.7539 \cdot 10^{29}$	$4.0123 \cdot 10^{30}$
$2.7000 \cdot 10^{-6}$	$7.4614 \cdot 10^{-5}$	$-1.3504 \cdot 10^{08}$	381 112	$+1.8505 \cdot 10^{29}$	$4.0027 \cdot 10^{30}$
$1.0000 \cdot 10^{-5}$	$2.0001 \cdot 10^{-4}$	$-3.9311 \cdot 10^{09}$	159 207	$+1.5139 \cdot 10^{30}$	$2.6739 \cdot 10^{30}$
$1.0000 \cdot 10^{-4}$	$3.2271 \cdot 10^{-4}$	$-4.1179 \cdot 10^{09}$	34 300	$+3.7540 \cdot 10^{30}$	$4.3371 \cdot 10^{29}$
$2.0000 \cdot 10^{-4}$	$3.4455 \cdot 10^{-4}$	$-4.1186 \cdot 10^{09}$	21 608	$+4.0894 \cdot 10^{30}$	$9.8329 \cdot 10^{28}$
$3.0000 \cdot 10^{-4}$	$3.5511 \cdot 10^{-4}$	$-4.1187 \cdot 10^{09}$	16 490	$+4.1779 \cdot 10^{30}$	$9.7997 \cdot 10^{27}$
$3.5000 \cdot 10^{-4}$	$3.5875 \cdot 10^{-4}$	$-4.1187 \cdot 10^{09}$	14 879	$+4.1875 \cdot 10^{30}$	$2.1339 \cdot 10^{26}$
$3.5936 \cdot 10^{-4}$	$3.5936 \cdot 10^{-4}$	$-4.1187 \cdot 10^{09}$	14 620	$+4.1877 \cdot 10^{30}$	0
$3.6500 \cdot 10^{-4}$	$3.5971 \cdot 10^{-4}$	$-4.1187 \cdot 10^{09}$	14 469	$+4.1875 \cdot 10^{30}$	$7.4788 \cdot 10^{25}$

This table shows how the perfect synchronism condition with angular speed ω_2, represents a situation possible only in the past, and at the same time indicates that at such speed the system energy reaches a maximum and therefore an instable equilibrium configuration.
Also in this table we added one more line, not correct from a physical and mathematical point of view, with the only scope of showing that the total energy has effectively a maximum for $\omega_r = \omega_2$.
If the Earth-Moon system evolution would have been started from such instable equilibrium configuration, then a small perturbation of the system would have caused that the tidal action, in alternative to the present trend towards a future synchronization, could have determined a fast progressive approaching of the two bodies till a destructive crash, in the same way that a bicycle, stationary in vertical position, can fall indifferently to left or to right.
It is interesting to notice that the Earth-Moon distance varies very fast when the system approaches the maximum, around 4 billion years ago.

8 Limits of the model

Since the beginning I stressed the fact that the model described in this paper is necessarily approximated and therefore the conclusions could be easily put under discussion.
Nevertheless it is a great comfort to me that also in the literature on this subject there are very similar conclusions, either qualitatively or quantitatively, even if these results were obtained by mathematical models and calculations surely more complex than mine.
On the other hand the main merit of this model, in addition to the fact of being founded on conservation laws which have always been the most reliable and undiscussed principles of the physics[10], is the extreme simplicity of the equations, obtained after some convenient algebraic transformations, which can be easily solved by analytical methods.
In fact analytical solutions, unlike numerical ones, give a better understanding of the phenomena, as well as a direct knowledge of the mathematical relations between the solution and the physical quantities and parameters which the phenomenon is supposed to depend on.
But a particularly critic limitation of this model is represented by the simplifying assumption quoted at point 5 of paragraph 1.2, where it was said that inertial forces, present in the developing of the phenomenon, wouldn't be kept into consideration as usual, hypothesizing that the evolution of the system happened through a sequence of stationary states of equilibrium between the gravitational forces and the centrifugal reactions. These states were also characterized by the fact that the orbits of both the bodies remained always rigorously circular.
Such simplification should not have, in my opinion, severe consequences in the case of the evolution of an Earth-Moon type system, which seems destined to a perfect synchronism with orbits approximately once and half as greater as the present ones, but it could have unacceptable consequences in situation in which the initial conditions of total energy and angular momentum, as well as masses and inertial moments of the two bodies, would have values very different from the present case.
In other words the hypothesis of always circular orbits makes mathematically impossible the occurrence of an undefined mutual going away of the two bodies because in such case the angular momentum due to the revolutionary motion should become infinite, while on the contrary it must remain constant as the system is assumed isolated.
In fact, even if the orbital angular speed decreases with the increasing of the distance in accordance with relation (7), nevertheless from (9) it is easy to show that the angular momentum associated to the orbital motion is globally growing with r and is in particular directly proportional to $\sqrt{r}$, as such angular momentum is given by the following relation:

$$Q' = m_r \sqrt{\Sigma\, r}$$

Now this moment could never be compensated (i.e. remain finite), by the moments due to the proper rotational motions of the two bodies, because in this case the total energy should increase indefinitely for the contributions due to such motions, while we know that the total energy must always decrease for tidal friction effects.
Therefore the model brings to the conclusion that the system in all those cases in which it doesn't end with its own autodestruction because of the progressive approaching and collision between the two bodies, shall reach in any case a configuration of perfect synchronism in which the two bodies will rotate around the centre of mass as an only rigid body, while it is excluded the possibility of a third alternative of an always increasing separation between the two bodies.
Besides this conclusion is immediately evident from the diagram of the function $Q(\omega)$ in the situation of perfect synchronism represented by (24): if $Q_0 > Q_{min}$ then there always exist two real intersections of such diagram with the straight line $Q = Q_0$ and therefore two angular speeds for which such condition is satisfied.
In the physical reality of the phenomenon this indefinite going away couldn't be excluded a priori, if we suppose that the bodies go away describing spiral-shaped trajectories; in fact in this case the

[10] There were some doubts in the Thirties about the validity of the conservation laws when it seemed that the β decay didn't respect such principles: the question was overcome with the hypothesis of a new particle, the neutrino, whose existence was confirmed experimentally only 20 years later!

angular momentum due to the revolutionary motion could keep finite because, together with the decreasing of the orbital angular speed, the moment arm respectively to the origin O, would grow less than in the circular orbit case.
In order to check such possibility it would be necessary to make the physico-matematical model more complex with the introduction of the inertial forces and the elimination of the simplificative assumption of circular orbits.
Moreover it is reasonably foreseeable that the so obtained equations wouldn't be more solvable in analytical form, but only by numerical methods.

9 Conclusions

The model described in this paper accounts for the tendency of the Earth-Moon system to stabilize completely reaching the condition of a perfect synchronism under the action of the tidal forces.
Such action could equally be utilized in order to explain the formation of a system of planets rotating around a star starting from the condensation of a stellar gas cloud provided with rotatory motion.
Under the action of the gravitational forces the gas of the cloud would tend to concentrate in several amassments, each of them having its own mass and its own rotational speed around its symmetry axis. The main and more massive of these amassments, situated the more closer to the centre of mass of the whole cloud the more percentually greater is its mass in front of the total mass, will give origin to the star, while those of minor mass will give origin to the planets.
The arrangement of these planets at different distances on the respective orbits would therefore depend, besides the original positions where the amassments have been formed, on the reciprocal tidal actions, which in their turn would depend on the masses, the inertial moments and the rotational velocity around its own axis and on the composition and physical form of the cloud.
A part of these amassments could also not give origin to planets, but be absorbed by the central star always because of the tidal actions.
Finally more interesting could be examining if these tidal forces have had an important influence also in the formation of the galaxies and of their amassments and can give a better understanding of the galactic dynamics.

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References

[1] Claudio Beccari, *LaTeX, Guida ad un sistema di editoria elettronica*, Editore Ulrico Hoepli, Milano, 1991