Physics in the Space of the Inertial Local Frames.
(preliminary version)

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0.1 Introduction

These notes contain a revised and updated account of a series of articles published by the author and collaborators in the last three decades [1–24], concerning the physical applications of the space of inertial local frames.

In the standard approach, summarized in Chapter 1, one starts from a 4-dimensional pseudo-Riemannian spacetime $\mathcal{M}$ and builds, by means of a standard mathematical procedure [25–27], the 10-dimensional principal fiber bundle $\mathcal{S}$ of the Lorentz frames, which is a very useful, but not necessary, instrument for the treatment of the known relativistic theories of gravitation. Also Maxwell and Yang-Mills theories can be included in this scheme by considering an "extended" principal fibre bundle $\mathcal{S}_n$ with a higher dimension.

There is no need for a new detailed exposition of this argument, which has been treated by many authors. We only present in Chapter 1 the basic ideas, in order to introduce some definitions and notations and to write some important formulas to be used later.

Our main purpose is to present a substantially different point of view, namely we start from the manifold $\mathcal{S}$ with a direct physical interpretation, and, if some conditions to be physically verified are satisfied, we build the spacetime manifold $\mathcal{M}$ by means of a suitable mathematical procedure. If these conditions are not satisfied, we have a nonlocal theory. In this way, we can introduce in a classical geometry a fundamental length ℓ, which is suggested by quantum gravity.

The development of this alternative point of view gives the opportunity for a discussion of some general aspects of physics concerning, for instance, the relativity principle, symmetry transformations and conserved quantities. We shall try to give some indications of this kind whenever we think that it may be useful.

We restrict our attention to the classical aspects of geometry, with some applications to quantum theories of matter in a classical geometric background. Since we are not interested in nonrelativistic mechanics, by "classical" we always mean "nonquantum". We indicate, whenever it is necessary, the points in which a classical geometry is not consistent with quantum theory. We hope that a new approach to classical geometry can provide new starting points and new ideas for the construction of a quantum theory of gravitation, namely a quantum geometry.

In Chapter 2 we treat the geometry of the space $\mathcal{S}$, defined in terms of transformations, namely mappings of $\mathcal{S}$ into itself, interpreted as physical procedures that have the purpose of building a new frame starting from a preexistent one. The transformations have a strict operational interpretation, while the frames, namely the single points of $\mathcal{S}$, are not operationally defined. This remark provides a foundation for a general formulation of the relativity principle, that states that all the frames are, a priori, physically equivalent.

We dedicate some attention to the problems raised by the very concept of inertial local frame, which has to be defined in terms of some material objects which inavoidably interact with the objects under investigation and with the measuring instruments. This remark is connected with the difficulties encountered in the construction of a quantum theory of gravitation.

In Chapter 3 we introduce in the tangent spaces of $\mathcal{S}$ a cone that characterizes the infinitesimal "feasible" transformations. The symmetry group of this cone is $GL(4, \mathbf{R})$ and we think that it is not an accident that 4 also is the number of components of the Dirac fields that describe matter in the Standard Model of elementary particles. We also discuss some mathematical properties of this group, of some of its subgroups and of some of their representations. $GL(4, \mathbf{R})$ has a subgroup isomorphic to $SL(2, \mathbf{C})$ the universal covering of the proper orthochronous Lorentz group. We advance the idea that a field theory may have a spontaneouly broken symmetry with respect to a larger subgroup of $GL(4, \mathbf{R})$.

In Chapter 4 we deal with a Lagrangian approach to the classical field theories defined on $\mathcal{S}$ and with the connection between symmetries and conservation laws (Noether's theorem). In Chapter 5 we apply this formalism to several classical field theories usually defined on the spacetime manifold. In Chapter 6 we treat a scalar-tensor theory, giving a geometrical interpretation to the scalar field that replaces the gravitational constant.

In Chapter 7 we look for Lagrangian field theories with a symmetry group larger than the Lorentz group. This is a rather difficult problem and only preliminary, not completely satisfactory results are presented. We hope to give more complete results in a future version of these notes.

In Chapter 8 we describe the motion of a test particles by associating to it a set of frames that form a submanifold of $\mathcal{S}$. If we consider the particle as a small region in which the fields are particularly strong, the particle dynamics is provided by the balance equations of the underlying field theory. Alternatively, one can introduce an independent Lagrangian or Hamiltonian particle dynamics. The treatment of test particles is relevant for the physical interpretation of field theories.

The Chapters 9, 10 and 11 are not yet complete and contain only some references to the original papers. In the following versions they will present some other aspects of the formalism based on local inertial frames.

The present notes contain mainly already published material, though many ideas and calculations have been clarified and improved. We hope that a consistent description of the state of the art will be useful for a further progress. We devote a special attention to the best choice of the notations and conventions, modifying in some cases the choices adopted in the original articles. We cite the most relevant contributions of other authors, also if they do not agree completely with our point of view, but we do not claim that the list of references is complete. It will be improved in the following versions of these notes.

We do not try to give a set of references to the wide literature on quantum gravity. Even giving some references to review articles and books would imply a relative evaluation of the different approaches, which is neither necessary nor useful for the purposes of the present notes. In any case, an extended analysis of the classical theory is an important step for the construction of a quantum theory.

0.2 Notations and conventions

For the velocity of light, we always use the convention $c = 1$, while we write the Planck constant $\hbar$, the gravitational constant G and the fundamental length ℓ explicitly. We use rationalized units, namely a factor 4π appears in the Coulomb law, but not in the source terms of Maxwell's equations.

The indices i, j, k, l, m, n, p, q take the values $0, 1, 2, 3$ and label, for instance, the anholonomic components of the $SO(1, 3)$ tensors. The indices $\lambda, \mu, \nu, \sigma, \tau$ take the same values and label the local coordinates in the space-time $\mathcal{M}$ and the holonomic tensor components. The indices r, s, t take the values $1, 2, 3$. The indices u, v, w, x, y, z take the values $0, 1, 2, 3, 4$ and appear in the tensor calculus of the anti-de Sitter group $SO(2, 3)$. When explicitly stated, they also take the value 5 and are used in the tensor calculus of the group $SO(3, 3)$.

The indices $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta$ take the values $0, \ldots, 9 + n$, where n is the dimension of the internal gauge group. If $n > 0$, the indices a, b, c, d take the values $10, \ldots, 9 + n$ and label a basis in the Lie algebra of the internal gauge group. If $n = 1$, the index 10, which labels the generator of the electromagnetic gauge transformations, is replaced by the symbol $\bullet$ for typographical reasons.

The summation over a pair of repeated, upper and lower, indices is understood unless a different indication is given.

The Minkowskian metric tensor $g_{ik} = g^{ik}$ is diagonal and we put $g_{00} = -1$, $g_{rs} = \delta_{rs}$. This convention, different from the one adopted in the preceding articles, is particularly convenient in connection with Hamiltonian mechanics. If we indicate by p_i the generators of the active spacetime translations, conjugate to the Minkowskian spacetime coordinates x^i, the three quantities $p^r = p_r$ are the momentum components and $p^0 = -p_0$ is the generator of the passive time translations, namely the energy. Another advantage is that the Dirac γ matrices in the Majorana representation are real.

We indicate by ϵ the antisymmetric Levi-Civita symbol, with the normalization $\epsilon_{0123} = -\epsilon^{0123} = 1$ in relativistic tensor calculus and $\epsilon_{1234} = \epsilon^{1234} = 1$ in Dirac spinor calculus.

For the sign conventions concerning the Riemann curvature tensor, we follow ref. [28], where the conventions used by other authors are also discussed.

The three-dimensional vectors are indicated by bold-face letters. The scalar product, the vector product and the norm are represented respectively by $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ and $\|\mathbf{u}\|$. If $u = (u^0, \mathbf{u})$ and $v = (v^0, \mathbf{v})$ are four-vectors, we write their scalar product in the form $u \cdot v = g_{ik} u^i v^k = \mathbf{u} \cdot \mathbf{v} - u^0 v^0$.

The indices of Dirac spinors and γ matrices are usually understood, since we use a matrix notation. When it is necessary, we use for them the capital letters $A, B, C, D, \ldots$ that take the values $1, \ldots, 4$. In a similar way, the components of the nongeometric fields are represented by a one-column matrix and when necessary they are labelled by the capital letters U, V, W.

Modifying a convention used in the preceding articles, we assume that the structural group (for instance the Lorentz group) has a right action on the principal bundle, in agreement with the majority of the textbooks of differential geometry. The elements of the structural group act on the local frames, namely they have a passive interpretation. When they operates on the observables, namely they are considered from the active point of view, they have a left action, as it is usually assumed.

We use *italic fonts* to indicate important concepts that appear for the first time and can be found in the Index.

0.3 Some useful identities

In this Section we collect some identities that we shall often use in the calculations. They concern mainly the antisymmetric tensor ϵ^{ijkl}, the differential 1-forms ω^i and the Dirac matrices γ^i.

A first set of identities is

$$\epsilon^{ijkl}\epsilon_{ijkl} = -24, \qquad \epsilon^{mijk}\epsilon_{nijk} = -6\delta^m_n, \qquad \epsilon^{ijmn}\epsilon_{klmn} = -2(\delta^i_k\delta^j_l - \delta^j_k\delta^i_l),$$
$$\epsilon^{ijkl}\epsilon_{imnp} = -\delta^j_m\delta^k_n\delta^l_p - \delta^l_m\delta^j_n\delta^k_p - \delta^k_m\delta^l_n\delta^j_p + \delta^k_m\delta^j_n\delta^l_p + \delta^l_m\delta^k_n\delta^j_p + \delta^j_m\delta^l_n\delta^k_p. \quad (1)$$

Since an expression completely antisymmetric with respect to 5 indices that can take only 4 values must vanish, we have the useful identity

$$\epsilon_{ijkl}x_m - \epsilon_{mjkl}x_i - \epsilon_{imkl}x_j - \epsilon_{ijml}x_k - \epsilon_{ijkm}x_l = 0. \quad (2)$$

The differential forms

$$\eta = \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 = (24)^{-1}\epsilon_{ijkl}\omega^i \wedge \omega^j \wedge \omega^k \wedge \omega^l, \quad (3)$$

$$\eta_i = 6^{-1}\epsilon_{ijkl}\omega^j \wedge \omega^k \wedge \omega^l = i(A_i)\eta \quad (4)$$

appear in many formulas. They have the properties

$$\omega^i \wedge \omega^j \wedge \omega^k \wedge \omega^l = -\epsilon^{ijkl}\eta, \qquad \omega^j \wedge \omega^k \wedge \omega^l = -\epsilon^{ijkl}\eta_i, \qquad \omega^k \wedge \eta_i = \delta^k_i \eta. \quad (5)$$

The Dirac matrices, characterized by the equation

$$\gamma^i\gamma^k + \gamma^k\gamma^i = 2g^{ik}, \quad (6)$$

have the properties

$$\mathrm{Tr}\,(\gamma^i\gamma^k) = 4g^{ik}, \qquad \mathrm{Tr}\,(\gamma^i\gamma^j\gamma^k\gamma^l) = 4(g^{ij}g^{kl} - g^{ik}g^{jl} + g^{il}g^{jk}), \quad (7)$$

$$\gamma_5 = -\gamma^5 = \gamma_0\gamma_1\gamma_2\gamma_3 = -(24)^{-1}\epsilon_{ijkl}\gamma^i\gamma^j\gamma^k\gamma^l, \qquad (\gamma_5)^2 = -1, \quad (8)$$

$$\mathrm{Tr}\,(\gamma_5\gamma^i\gamma^j\gamma^k\gamma^l) = -4\epsilon^{ijkl}, \quad (9)$$

$$(\gamma^i\gamma^k - \gamma^k\gamma^i)\gamma_5 = \epsilon^{ik}{}_{jl}\gamma^j\gamma^l. \quad (10)$$

$$\gamma_i\gamma_j\gamma_k = \epsilon_{lijk}\gamma^l\gamma_5. \quad (11)$$

Chapter 1

The (extended) principal fiber bundle of the Lorentz frames

1.1 Tetrads

With the aim of introducing some concepts and notations, we consider first a theory based on a pseudo-Riemannian connected 4-dimensional spacetime manifold $\mathcal{M}$. In order to define the components of vector and tensor fields, we have to define a basis in the tangent spaces $T_x\mathcal{M}$ at all the points $x \in \mathcal{M}$. We can introduce local coordinates x^μ and the basis provided by the vectors $\partial_\mu = \partial/\partial x^\mu$ (remember that there is a one-to-one correspondence between vector fields and first order linear differential operators). We obtain in this way the *holonomic components*.

In particular, we indicate by $g_{\mu\nu}$ the holonomic components of the covariant metric tensor and by $g^{\mu\nu}$ the elements of the inverse matrix, which are the components of a contravariant tensor. These tensors can be used to raise and lower the holonomic indices of other tensors.

We can also introduce at every point $x \in \mathcal{M}$ an orthonormal *tetrad* (also called, in German, *Vierbein*) of four-vectors $e_i(x)$, with the property

$$e_i \cdot e_k = g_{\mu\nu} e_i^\mu e_k^\nu = g_{ik}. \tag{1.1}$$

By means of these bases we obtain the *anholonomic components* of vector and tensor fields. An explanation of the notations and the values of the constant anholonomic components $g_{ik} = g^{ik}$ of the metric tensor, which can be used to raise and lower the anholonomic indices, is given in Section 0.2. The use of a tetrad field (also called *moving frame* or *repère mobile* in French) is a powerful instrument in differential geometry, extensively used by E. Cartan [29].

We assume that the reader is acquainted with the simpler aspects of Riemannian geometry in the holonomic formalism (see for instance [30–32]) and in this Section we present some basic concepts of the anholonomic formalism, necessary for the motivation of the more general scheme introduced in Section 2.

The matrices e_i^μ and their inverses e_μ^i can be used to transform holonomic indices into anholonomic ones and vice-versa, namely to perform a change of basis. In particular, the formula

$$g_{\mu\nu} = e_\mu^i e_\nu^k g_{ik} \tag{1.2}$$

shows that the tetrads determine the metric tensor. The quantities e_μ^i are the components of a dual tetrad of covariant four-vectors e^i that form a basis in the cotangent space $T_x^*\mathcal{M}$.

One can show that the set of all the tetrads is a differentiable manifold, which we indicate by $\mathcal{S}$. The tetrads with a common origin x form a *fiber* and one can consider the differentiable *projection mapping* $\pi : \mathcal{S} \to \mathcal{M}$ associating to every tetrad $s \in \mathcal{S}$ its origin $x = \pi(s) \in \mathcal{M}$.

If $\Lambda^i{}_k$ is a 4×4 Lorentz matrix, it transforms a tetrad into another tetrad according to the formula

$$e_i \to e'_i = e_k \Lambda^k{}_i, \qquad e^i \to e'^i = (\Lambda^{-1})^i{}_k e^k, \tag{1.3}$$

which can also be written in the abbreviated form $s \to s' = s\Lambda$. We see that there is a right action of the Lorentz group $\mathcal{L}$ on the manifold $\mathcal{S}$. This means that $s(\Lambda\Lambda') = (s\Lambda)\Lambda'$ and the parentheses are not necessary. The action of $\mathcal{L}$ preserves the fibers (namely it commutes with π) and acts freely and transitively on every fiber. As a consequence, every fiber is diffeormorphic to $\mathcal{L}$. In this situation one says [25–27] that $\mathcal{S}$ is a *principal fiber bundle* with *base* $\mathcal{M}$ and *structural group* $\mathcal{L}$.

There is an important detail to be clarified. It is natural to assume that in every tangent space $T_x\mathcal{M}$ it is possible to choose in a continuous way a future cone, namely that $\mathcal{M}$ is *time orientable* [33]. This is a physically justified restriction to the topology of $\mathcal{M}$. Then it is natural to consider only tetrads with the timelike four-vector e_0 belonging to the future cone. As a consequence, $\mathcal{L}$ has to be the orthochronous Lorentz group.

The group $\mathcal{L}$ contains the space inversion and has two connected components. Of course, the fibers have the same property. If the connected manifold $\mathcal{M}$ is *orientable*, $\mathcal{S}$ has two connected components containing the left-handed and the right-handed tetrads. Otherwise, $\mathcal{S}$ is connected.

A principal fiber bundle is the natural arena for *gauge field theories*. In particular it has been shown [34–38] that General Relativity and other theories of gravitation can be formulated as gauge theories of the Lorentz group $\mathcal{L}$ or of the Poincaré group $\mathcal{P}$.

1.2 Tensor and spinor fields

The anholonomic components of tensor fields on $\mathcal{M}$ are uniquely determined when the frame $s \in \mathcal{S}$ is given and have to be considered as scalar fields on $\mathcal{S}$. They behave in a particular way when s moves on a fiber. For instance, a scalar (on $\mathcal{M}$) field S has the property

$$S(s\Lambda) = S(s), \qquad \Lambda \in \mathcal{L}, \tag{1.4}$$

namely it is constant on the fibers. The anholonomic components V^i of a vector field satisfy the condition

$$V^i(s\Lambda) = V \cdot e^i(s\Lambda) = V \cdot ((\Lambda^{-1})^i{}_k e^k(s)) = (\Lambda^{-1})^i{}_k V^k(s). \tag{1.5}$$

The inverse matrix Λ^{-1} appears because Λ is interpreted a *passive transformation*, namely a change of the frame leaving the vector V unchanged. If, instead, we consider an *active transformation*, that changes the vector V leaving the frame unchanged, the matrix Λ appears in the transformation property.

A more general tensor field is characterized by the condition

$$\Psi(s\Lambda) = \Sigma(\Lambda^{-1})\Psi(s), \tag{1.6}$$

where the elements of the one-column matrix Ψ are the anholonomic tensor components and the square matrix Σ belongs to a linear representation of $\mathcal{L}$.

A similar formula holds for a *spinor*, but Σ is a two-valued representation and the spinor components too are two-valued functions of s. A more rigorous approach is to consider a double covering $\tilde{\mathcal{S}}$ of $\mathcal{S}$, which is a principal fiber bundle with structural group $\tilde{\mathcal{L}}$, a double covering of $\mathcal{L}$, which contains $SL(2, \mathbf{C})$ and two elements corresponding to the space inversion [39]. Then the components of the spinor fields are one-valued functions on $\tilde{\mathcal{S}}$ and Σ is a one-valued linear representation of $\tilde{\mathcal{L}}$. The fiber bundle $\tilde{\mathcal{S}}$ exists only if $\mathcal{M}$ has suitable topological properties and in this case one says that $\mathcal{M}$ admits a *spin structure*. The use of tetrads to treat spinor fields on a curved spacetime has been introduced by H. Weyl [40].

1.3 Infinitesimal Lorentz transformations

For many purposes, it is useful to consider infinitesimal Lorentz transformations of the kind

$$\Lambda^i{}_k \sim \delta^i_k + \zeta^i{}_k, \qquad \zeta^{ik} = -\zeta^{ki}. \tag{1.7}$$

Their action on $\mathcal{S}$ is generated by the vector fields (or differential operators) $A_{[ik]}$ defined by

$$f(s\Lambda) \sim f(s) + 2^{-1}\zeta^{[ik]}A_{[ik]}f(s), \tag{1.8}$$

where f is an arbitrary differentiable function and Λ is given by eq. (1.7).

These vector fields are tangent to the fibers, which are diffeomorhic to $\mathcal{L}$. They can be considered as generators of right translations of the group $\mathcal{L}$ and they define a basis of its Lie algebra $o(1,3)$. Here and in the following, in order to obtain more readable formulas, we always enclose into square brackets the pairs of antisymmetric indices which label the elements of this basis.

The sign of the parameters $\zeta^{[ik]}$ depends on the choice of the metric g_{ik}. With our choice (see sect. 0.2), for instance, $\zeta^{[21]} = \zeta^2{}_1$ describes a counterclockwise rotation around e_3 and $\zeta^{[03]} = \zeta^0{}_3$ describes a boost along e_3.

If we introduce in an open region of $\mathcal{M}$ a set of local coordinates x^μ, we can parametrize locally $\mathcal{S}$ by means of the redundant coordinates (x^μ, e^μ_i) constrained by eq. (1.1). Then we can write the explicit formula

$$A_{[ik]} = (g_{kj}e^\mu_i - g_{ij}e^\mu_k)\frac{\partial}{\partial e^\mu_j}. \tag{1.9}$$

One has to be careful in dealing with partial derivatives with respect to variables which are not independent. First we define the vector fields (1.9) in the space of the unconstrained variables and then we check that they are tangent to the manifold defined by eq. (1.1) namely that, on the same manifold, we have

$$A_{[ik]}(g_{\mu\nu}e^\mu_j e^\nu_l) = 0. \tag{1.10}$$

By computing the commutator we obtain

$$[A_{[ik]}, A_{[jl]}] = 2^{-1}\hat{F}^{[mn]}_{[ik][jl]}A_{[mn]}, \tag{1.11}$$

where the quantities

$$\begin{aligned}\hat{F}^{[mn]}_{[ik][jl]} = \delta^m_i g_{kj}\delta^n_l - \delta^m_k g_{ij}\delta^n_l - \delta^m_i g_{kl}\delta^n_j + \delta^m_k g_{il}\delta^n_j \\ -\delta^n_i g_{kj}\delta^m_l + \delta^n_k g_{ij}\delta^m_l + \delta^n_i g_{kl}\delta^m_j - \delta^n_k g_{il}\delta^m_j\end{aligned} \tag{1.12}$$

are the structure constants of the Lorentz Lie algebra.

The behavior of tensor and spinor fields under Lorentz transformations, described in Section 1.2 can be written as a differential equations. For the scalar and vector fields we obtain

$$A_{[ik]}S = 0, \qquad A_{[ik]}V^j = -(\delta^j_i g_{kl} - \delta^j_k g_{il})V^l \tag{1.13}$$

and in general, if we write the components of a tensor or spinor in the form of a one-column matrix Ψ, we have

$$A_{[ik]}\Psi = -\Sigma_{[ik]}\Psi. \tag{1.14}$$

The square matrices $\Sigma_{[ik]}$ are defined by

$$\Sigma(\Lambda) \sim 1 + 2^{-1}\zeta^{[ik]}\Sigma_{[ik]}, \tag{1.15}$$

where Λ is given by eq. (1.7).

They form a representation of the Lie algebra $o(1,3)$ of $\mathcal{L}$, namely we have

$$[\Sigma_{[ik]}, \Sigma_{[jl]}] = 2^{-1}\hat{F}^{[mn]}_{[ik][jl]}\Sigma_{[mn]}. \tag{1.16}$$

For a Dirac spinor we have

$$\Sigma_{[ik]} = 2^{-2}(\gamma_i\gamma_k - \gamma_k\gamma_i). \tag{1.17}$$

$$\Sigma_{[ik]}\gamma_j - \gamma_j\Sigma_{[ik]} = g_{kj}\gamma_i - g_{ij}\gamma_k. \tag{1.18}$$

The properties of the γ-matrices are summarized in Section 0.3.

1.4 Maxwell and Yang-Mills fields

As the theories of gravitation, also the gauge field theories with internal gauge group $\mathcal{G}$ [41] have an elegant geometric treatment in the framework of a principal fibre bundle with structural group $\mathcal{G}$ [42]. We always specify "internal" because gravitation too is described by a gauge theory. If $\mathcal{G} = U(1) = SO(2)$, we obtain Maxwell's electromagnetism and for $\mathcal{G} = SU(2)$ we have the original Yang-Mills theory [43].

Several authors [44–46] have proposed a unified treatment of gravitation and internal gauge theories based on a principal fibre bundle with base $\mathcal{M}$ and structural group $\mathcal{L} \times \mathcal{G}$. If $\mathcal{G}$ is a Lie group with dimension n, this bundle has dimension $10 + n$. We call it the *bundle of extended frames* and we indicate it by $\mathcal{S}_n$. It can also be considered as a principal fibre bundle with base $\mathcal{S}$ and structural group $\mathcal{G}$. Of course, if $n = 0$ we have $\mathcal{S}_0 = \mathcal{S}$. This approach is similar to the Kaluza-Klein unification of gravitation and electromagnetism [47, 48], but it is conceptually rather different.

The right action of $\mathcal{G}$ on $\mathcal{S}_n$ is generated by n vector fields A_a, where the index a labels a basis of the Lie algebra of $\mathcal{G}$. In the treatment of the Maxwell field, we have $n = 1$ and we indicate the generator of the electromagnetic gauge transformations by $A_\bullet$. If $n > 1$, the vector fields A_a satisfy the commutation relations (or Lie brackets)

$$[A_{[ik]}, A_a] = 0, \qquad [A_a, A_b] = \hat{F}^c_{ab} A_c, \tag{1.19}$$

where $\hat{F}^c_{ab}$ are the structure constants of the Lie algebra of $\mathcal{G}$.

In order to obtain a local parametrization of $\mathcal{S}_n$, we have to choose, besides a local coordinate system in $\mathcal{M}$, a gauge at every point $x \in \mathcal{M}$. Then the extended frame $s \in \mathcal{S}_n$ is determined by the quantities (x^μ, e_i^μ, g), where $g \in \mathcal{G}$, represents the gauge transformation from the conventionally chosen gauge at $x = \pi(s)$ to the gauge choice at s. The group element g, in turn, can be locally parametrized by n real coordinates. Note that g is not affected by the right action of the Lorentz group $\mathcal{L}$. The generators A_a of the internal gauge transformations also describe the infinitesimal right translations of the group $\mathcal{G}$ and there is no problem in using the same symbols for the vector fields defined in $\mathcal{S}_n$ and in $\mathcal{G}$.

In Section 1.5 we need the vector fields A_a^L that generate the left translations on the group $\mathcal{G}$. They commute with the generators A_a of the right translations and satisfy the commutation relations

$$[A_a^L, A_b^L] = -\hat{F}^c_{ab} A_c^L \tag{1.20}$$

(note the minus sign). They can be written in the form

$$L_a^L = D^b{}_a(g^{-1})A_b, \tag{1.21}$$

where the matrices $D^b{}_a$ belong to the adjoint representation of $\mathcal{G}$. The vector fields A_a^L, originally defined on $\mathcal{G}$, can also be considered as vector fields on $\mathcal{S}_n$, but in this case they depend on the choice of the parametrization.

In Section 1.8 we use the left invariant *Maurer-Cartan one-forms* χ^b on the gauge group $\mathcal{G}$ defined by the formula

$$\chi^b(A_a) = \delta_a^b. \tag{1.22}$$

They can be written in terms of the local coordinates of the group $\mathcal{G}$ and they can also be considered as differential forms defined on $\mathcal{S}_n$, which, however depend on the choice of the parametrization. In any case we have [25, 26]

$$d\chi^a = -2^{-1}\hat{F}^a_{bc}\,\chi^b \wedge \chi^c. \tag{1.23}$$

The group $\mathcal{G}$ acts linearly on the fields. If, as in eq. (1.14), we consider the field components as the elements of a one-column matrix Ψ, we have

$$\Psi(sg) = \Sigma(g^{-1})\Psi \tag{1.24}$$

and the infinitesimal transformations are given by

$$A_a\Psi = -\Sigma_a\Psi. \tag{1.25}$$

The matrices Σ_a form a representation of the Lie algebra of $\mathcal{G}$, namely

$$[\Sigma_a, \Sigma_b] = \hat{F}^c_{ab}\Sigma_c. \tag{1.26}$$

We use the same symbol Σ for both the representations of $\mathcal{L}$ and of $\mathcal{G}$, because we consider them as special cases of a representation of the structural group $\mathcal{L} \times \mathcal{G}$.

If $\mathcal{G} = U(1)$, g is a phase factor and it is convenient to put $g = \exp(ie\varphi)$, where φ is a cyclic real parameter with period $2\pi e^{-1}$ and e is the elementary electric charge. In this case we have

$$A_\bullet^L = A_\bullet = \frac{\partial}{\partial\varphi}, \qquad \chi^\bullet = d\varphi. \tag{1.27}$$

If Ψ is a complex field carrying the electric charge Ze, we have

$$A_\bullet\Psi = \frac{\partial}{\partial\varphi}\Psi = -\Sigma_\bullet\Psi = -iZe\Psi. \tag{1.28}$$

1.5 Parallel transport

A fundamental concept in Riemannian geometry is the *parallel transport*. If a tetrad vector e_j is parallel transported from a point with coordinates x^λ to a point with coordinates $x^\lambda + dx^\lambda$, we have (as for any other contravariant four-vector field) [28, 30–32]

$$\delta e_j^\mu = -\Gamma^\mu_{\nu\lambda} e_j^\nu dx^\lambda, \tag{1.29}$$

where $\Gamma^\mu_{\nu\lambda}$ are the *connection coefficients*. They depend only on x^λ and satisfy the *metricity condition*

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \Gamma^\sigma_{\mu\lambda} g_{\sigma\nu} - \Gamma^\sigma_{\nu\lambda} g_{\mu\sigma} = 0, \tag{1.30}$$

that assures that the covariant derivative of the metric tensor vanishes. In a torsionless theory, the connection coefficients are given by the Christoffel symbols.

If $n > 0$, the action of a parallel displacement on the group element g is an infinitesimal left translation that depends linearly on dx^λ. In conclusion, a parallel displacement of the tetrads in the direction of the tetrad four-vector e_i is described by the vector field

$$A_i = e_i^\lambda \left(\frac{\partial}{\partial x^\lambda} - \Gamma^\mu_{\nu\lambda} e_j^\nu \frac{\partial}{\partial e_j^\mu} + a_\lambda^a A_a^L \right), \tag{1.31}$$

where $a_\lambda^a(x)$ are the potentials of the gauge field and A_a^L are the generators of the left translations introduced in Section 1.4. In this case too, since we are not dealing with independent variables, we have to verify that the fields (1.31) are tangent to the manifold defined by eq. (1.1), namely that

$$A_i(g_{\mu\nu} e_k^\mu e_j^\nu) = 0. \tag{1.32}$$

This is a consequence of the condition (1.30).

After some calculations, we find for the commutators the following expressions

$$[A_{[ik]}, A_j] = \hat{F}^l_{[ik]j} A_l, \tag{1.33}$$

$$[A_a, A_j] = 0, \tag{1.34}$$

$$[A_i, A_k] = 2^{-1} F_{ik}^{[jl]} A_{[jl]} + F_{ik}^j A_j + F_{ik}^a A_a, \tag{1.35}$$

where

$$\hat{F}^{l}_{[ik]j} = g_{kj}\delta^{l}_{i} - g_{ij}\delta^{l}_{k}, \tag{1.36}$$

$$F^{[jl]}_{ik} = -e^{\lambda}_{i}e^{\sigma}_{k}e^{j}_{\mu}e^{l\nu}R^{\mu}_{\nu\lambda\sigma}, \qquad R^{\mu}_{\nu\lambda\sigma} = \frac{\partial\Gamma^{\mu}_{\nu\sigma}}{\partial x^{\lambda}} - \frac{\partial\Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\tau\lambda}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\mu}_{\tau\sigma}\Gamma^{\tau}_{\nu\lambda}, \tag{1.37}$$

$$F^{j}_{ik} = e^{\lambda}_{i}e^{\sigma}_{k}e^{j}_{\mu}S^{\mu}_{\lambda\sigma}, \qquad S^{\mu}_{\lambda\sigma} = \Gamma^{\mu}_{\lambda\sigma} - \Gamma^{\mu}_{\sigma\lambda}, \tag{1.38}$$

$$F^{a}_{ik} = e^{\lambda}_{i}e^{\sigma}_{k}D^{a}{}_{d}(g^{-1})F^{d}_{\lambda\sigma}, \qquad F^{d}_{\lambda\sigma} = \frac{\partial a^{d}_{\sigma}}{\partial x^{\lambda}} - \frac{\partial a^{d}_{\lambda}}{\partial x^{\sigma}} - \hat{F}^{d}_{bc}a^{b}_{\lambda}a^{c}_{\sigma}. \tag{1.39}$$

The quantities $R^{\mu}_{\nu\lambda\sigma}$, $S^{\mu}_{\lambda\sigma}$ and $F^{d}_{\lambda\sigma}$ are, respectively, the holonomic components of the *Riemann curvature tensor*, the *torsion tensor* and the *gauge field strength*. The quantities $F^{[jl]}_{ik}$, F^{j}_{ik} and F^{a}_{ik} are, up to a sign convention, the anholonomic components of the same tensors, given as functions on $\mathcal{S}_n$.

The structure constants (1.36) can be used to write the Lorentz transformation properties of contravariant and covariant four-vectors in the form

$$A_{[ik]}V^{j} = -\hat{F}^{j}_{[ik]l}V^{l}, \qquad A_{[ik]}V_{j} = \hat{F}^{l}_{[ik]j}V_{l}. \tag{1.40}$$

1.6 Covariant derivatives and spin connection

If we consider the anholonomic components V^j of a vector field carrying a charge Ze, by means of the useful formula

$$\frac{\partial e^{k}_{\rho}}{\partial e^{\mu}_{j}} = -e^{k}_{\mu}e^{j}_{\rho}, \tag{1.41}$$

we obtain

$$A_{i}V^{k} = e^{\lambda}_{i}e^{k}_{\mu}\left(\frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\nu\lambda}V^{\nu} - iZea^{\bullet}_{\lambda}V^{\mu}\right), \tag{1.42}$$

namely the anholonomic components of the covariant derivatives of V. This result was expected, if one remembers the meaning of the vector fields A_i, and a similar result holds for the anholonomic components of any tensor or spinor field, also with nontrivial transformation properties under the gauge group $\mathcal{G}$. One may consider the differential operator A_i as the covariant derivative in the direction of e_i.

For some applications one needs an explicit expression of the kind (1.42) also in the general case. If spinor fields are involved, one has to introduce, besides a local coordinate system in $\mathcal{M}$, a tetrad field [40], namely to assign a tetrad $e_i(x)$ to the points of a region of spacetime. If one also assigns to every point a choice of the gauge, one obtains a *local section* $x \to s(x)$ of the fiber bundle $\mathcal{S}_n$. The existence of a global section is not assured. Of course, $\pi(s(x)) = x$.

The group element $(\Lambda, g) \in \mathcal{L} \times \mathcal{G}$ represent the element of the structural group that transforms $s(x)$ into s and we have

$$e_i = e_k(x)\Lambda^k{}_i, \qquad \Lambda^k{}_i = e^k_\mu(x)e^\mu_i, \tag{1.43}$$

$$\Phi(s) = \Sigma(\Lambda^{-1})\Sigma(g^{-1})\Phi(x), \qquad \Phi(x) = \Phi(s(x)). \tag{1.44}$$

The variables $(x^\lambda, \Lambda^k{}_i, g)$ provide a new parametrization of the elements $s \in \mathcal{S}$ and, by means of the formulas

$$\left(\frac{\partial}{\partial x^\lambda}\right)_e = \left(\frac{\partial}{\partial x^\lambda}\right)_\Lambda + \frac{\partial e^k_\nu(x)}{\partial x^\lambda} e^\nu_j \frac{\partial}{\partial \Lambda^k{}_j}, \qquad \frac{\partial}{\partial e^\mu_j} = e^k_\mu(x)\frac{\partial}{\partial \Lambda^k{}_j}, \tag{1.45}$$

we obtain from eq. (1.31)

$$A_i = e^\lambda_i \left(\left(\frac{\partial}{\partial x^\lambda}\right)_\Lambda - \Gamma^j_{k\lambda}\Lambda^k{}_l \frac{\partial}{\partial \Lambda^j{}_l} + a^a_\lambda A^L_a \right). \tag{1.46}$$

We have introduced the quantities

$$\Gamma^j_{k\lambda}(x) = e^j_\mu(x)e^\nu_k(x)\Gamma^\mu_{\nu\lambda}(x) + e^j_\mu(x)\frac{\partial e^\mu_k(x)}{\partial x^\lambda}, \tag{1.47}$$

namely the connection coefficients in the anholonomic basis, also called the *spin connection coefficients*. Note that the connection coefficients do not transform as the components of a tensor under a change of the basis in the tangent spaces. It follows from the metricity condition (1.30) that

$$\Gamma^k_{i\lambda}g^{ij} = \Gamma^{kj}_\lambda = -\Gamma^{jk}_\lambda = \Gamma^{[kj]}_\lambda. \tag{1.48}$$

By means of the formulas

$$\Lambda^k{}_l \frac{\partial \Sigma(\Lambda^{-1})}{\partial \Lambda^j{}_l} = -2^{-1}\Sigma(\Lambda^{-1})g^{ki}\Sigma_{[ji]}, \qquad A^L_a \Sigma(g^{-1}) = -\Sigma(g^{-1})\Sigma_a, \tag{1.49}$$

we finally obtain

$$A_i\Psi = \Sigma(\Lambda^{-1})\Sigma(g^{-1})e^\lambda_i \left(\frac{\partial \Psi(x)}{\partial x^\lambda} + 2^{-1}\Gamma^{[jk]}_\lambda \Sigma_{[jk]}\Psi(x) - a^a_\lambda \Sigma_a \Psi(x) \right). \tag{1.50}$$

The expression in the parenthesis is the covariant derivative on the section $s(x)$ and the preceding factors transform it into the covariant derivative in the direction of e_i at a generic point s.

From the commutation relation (1.33), we see that the covariant derivative has the correct Lorentz transformation property, namely

$$A_{[ik]} A_j \Psi = \hat{F}^l_{[ik]j} A_l \Psi - \Sigma_{[ik]} A_j \Psi. \tag{1.51}$$

1.7 A compact formalism

In the present Section we introduce some notations that permit us to write the formulas of the preceding Sections in a very compact form. In this way we simplify some calculations, but, at this stage, we do not introduce any new mathematical or physical idea. We also write all the relevant formulas using the concepts of differential geometry that do not refer to a particular local coordinate system or to a local section of the fiber bundle. In the following Chapter 2 we shall give a different, more general, interpretation of this compact formalism and use it to introduce new physical ideas.

We have seen that the (possibly extended) bundle of Lorent frames is a manifold $\mathcal{S}_n$ with dimension $10 + n$, where n is the dimension of an internal gauge group $\mathcal{G}$. We have also seen that its most important geometric properties are described by the vector fields A_i, $A_{[ik]}$ and, if $n > 0$, A_a. We use for all these fields a unified notation A_α, where α takes the values $0, \ldots, 9 + n$.

More precisely, the fields $A_0, \ldots, A_3$ generate parallel displacements of the tetrads along the directions of the tetrad vectors, $A_4 = A_{[32]}$, $A_5 = A_{[13]}$, $A_6 = A_{[21]}$ generate rotations around the spatial vectors of the tetrad, $A_7 = A_{[01]}$, $A_8 = A_{[02]}$, $A_9 = A_{[03]}$ generate Lorentz boosts along the same spatial vectors and $A_{10}, \ldots, A_{9+n}$ generate the infinitesimal transformation of the internal gauge group. If $n = 1$, in order to avoid a two digits index, we write $A_\bullet$ instead of A_{10}.

The vectors $A_\alpha(s)$, $\alpha = 0, \ldots, 9 + n$ are linearly independent and they provide a basis in every tangent space $T_s\mathcal{S}_n$, $s \in \mathcal{S}_n$. By means of this basis, one can identify in a natural way all the tangent spaces $T_s\mathcal{S}_n$ with a single $(10 + n)$-dimensional vector space $\mathcal{T}_n$.

The subspace of $T_s\mathcal{S}_n$ generated by the vectors $A_0(s), \ldots, A_3(s)$ is called the *horizontal subpace*, while the subspace generated by the vectors $A_4(s), \ldots,$ $A_{9+n}(s)$ is called the *vertical subpace*. The vertical subspaces are tangent to the fibers, while the horizontal subspaces define a *connection* in the principal bundle $\mathcal{S}_n$. We also consider these subspaces as subspaces of the vector space $\mathcal{T}_n$ and we indicate them, respectively, by $\mathcal{T}_H$ and $\mathcal{T}_V$. The last subspace can be identified with the Lie algebra of the structural group. If $n > 0$, the vertical subspace $\mathcal{T}_V$ is the direct sum of the subspace $\mathcal{T}_L$ generated by $A_{[ik]}$ and the subspace $\mathcal{T}_I$ generated by A_a.

The vector fields A_α can be considered as first order differential operators and their commutators (Lie brackets) can be written in the form

$$[A_\alpha, A_\beta] = F^\gamma_{\alpha\beta} A_\gamma. \tag{1.52}$$

The quantities $F^\gamma_{\alpha\beta} = -F^\gamma_{\beta\alpha}$, are called *structure coefficients*.

We also introduce in the space $\mathcal{S}_n$ the differential 1-forms ω^β, dual to the vector fields A_α, defined by

$$i(A_\alpha)\omega^\beta = \omega^\beta(A_\alpha) = \delta^\beta_\alpha, \tag{1.53}$$

where $i(X)$ is the interior product operator acting on the differential forms. Their exterior derivatives are given by

$$d\omega^\gamma = -2^{-1} F^\gamma_{\alpha\beta}\, \omega^\alpha \wedge \omega^\beta. \tag{1.54}$$

The exterior products of these 1-forms provide a basis in the space of differential forms of higher degree. We say that a term containing the product of d_H forms of the kind ω^i has *horizontal degree* d_H, a term containing the product of d_L forms of the kind $\omega^{[ik]}$ has *Lorentz vertical degree* d_L and a term containing the product of d_I forms of the kind ω^a has *internal degree* d^I. We use the notation (d_H, d_L, d_I) to describe the partial degrees of a term. The *total degree* is the sum of the partial degrees. These concepts are very useful in the calculations.

From the Jacobi identity satisfied by the commutators (1.52) or considering the vanishing exterior derivation of eq. (1.54), we find the generalized Jacobi identity

$$A_\alpha F^\delta_{\beta\gamma} + A_\beta F^\delta_{\gamma\alpha} + A_\gamma F^\delta_{\alpha\beta} - F^\eta_{\alpha\beta} F^\delta_{\eta\gamma} - F^\eta_{\beta\gamma} F^\delta_{\eta\alpha} - F^\eta_{\gamma\alpha} F^\delta_{\eta\beta} = J^\delta_{\alpha\beta\gamma} = 0. \tag{1.55}$$

We see from eqs. (1.11), (1.19), (1.33), (1.34) and (1.35) that the structure coefficients coincide with the structure constants $\hat{F}^\gamma_{\alpha\beta}$ of the Lie algebra of the extended Poincaré group $\mathcal{P} \times \mathcal{G}$, with the exception of the coefficients

F^{γ}_{ik}, which give the anholonomic components of the torsion, curvature and gauge field strength tensors defined by eqs. (1.37), (1.38) and (1.39).

The generalized Jacobi identity (1.55) represents, in a very compact form, a large number of physically relevant formulas. In particular:

- $J^a_{ijk} = 0$ is the homogeneous set of the Maxwell or Yang-Mills equations;
- $J^{[mn]}_{ijk} = 0$ is the Bianchi identity for the curvature;
- $J^l_{ijk} = 0$ is the Bianchi identity for the torsion or a symmetry property of the curvature if the torsion vanishes;
- $J^b_{aik} = 0$ represents the gauge transformation property of the Maxwell or Yang-Mills field strength;
- $J^{[mn]}_{aik} = 0$ represents the gauge invariance of the curvature;
- $J^l_{aik} = 0$ represents the gauge invariance of the torsion.
- $J^a_{[ij]lk} = 0$ represents the tensor nature of the Maxwell or Yang-Mills field strength;
- $J^{[mn]}_{[ij]lk} = 0$ represents the tensor nature of the curvature;
- $J^l_{[ij]lk} = 0$ represents the tensor nature of the torsion.

For other values of the indices we obtain the Jacobi identity for the structure constants of the extended Poincaré algebra.

1.8 Connection, soldering, curvature and torsion forms

If we introduce, as in Section 1.4, a local parametrization of $\mathcal{S}_n$, starting from the explicit formulas (1.9) and (1.31) for the vector fields A_α and from the definition (1.53), we can compute the following explicit expressions of the forms ω^β:

$$\omega^i = e^i_\lambda dx^\lambda. \tag{1.56}$$

$$\omega^{[ik]} = g^{kj} e^i_\mu (de^\mu_j + e^\nu_j \Gamma^\mu_{\nu\lambda} dx^\lambda), \tag{1.57}$$

$$\omega^a = \chi^a - D^a{}_b(g^{-1}) a^b_\lambda dx^\lambda, \tag{1.58}$$

where χ^a is the Maurer-Cartan form of $\mathcal{G}$ defined in Section 1.4. One can show by means of eq. (1.1) that the expression (1.57) is antisymmetric in the indices i, k. It is a useful exercise to show that eq. (1.54) is satisfied.

The one-forms $\omega^{[ik]}$ and ω^a are called the components of the *connection form*, which takes its values in $\mathcal{T}_V$, namely in the Lie algebra of the structural group. The one-forms ω^i are called the components of the *soldering form*, also called the *canonical form*, that takes its values in $\mathcal{T}_H$.

Other useful quantities defined in the literature on fibre bundles [25–28] are the *curvature form*, a two-form taking values in $\mathcal{T}_V$, with components

$$\Omega^{[ik]} = -2^{-1} F^{[ik]}_{jl} \omega^j \wedge \omega^l = d\omega^{[ik]} + 2^{-3} \hat{F}^{[ik]}_{[jl][mn]} \omega^{[jl]} \wedge \omega^{[mn]}, \tag{1.59}$$

$$\Omega^a = -2^{-1} F^a_{jl} \omega^j \wedge \omega^l = d\omega^a + 2^{-1} \hat{F}^a_{bc} \omega^b \wedge \omega^c, \tag{1.60}$$

and the *torsion form*, a two-form taking values in $\mathcal{T}_H$ with components

$$\Omega^i = -2^{-1} F^i_{jl} \omega^j \wedge \omega^l = d\omega^i + 2^{-1} \hat{F}^i_{[kl]j} \omega^{[kl]} \wedge \omega^j. \tag{1.61}$$

Note that these formulas, called *structure equations* agree with eq. (1.54).

1.9 Flat Minkowski spacetime and Poincaré group

It is interesting to consider with more detail the simple case in which $\mathcal{M}$ is the flat *Minkowski spacetime*. In a first treatment we disregard the internal gauge group. Then we can choose a distinguished local frame, namely a *distinguished point* $\hat{s} \in \mathcal{S}$, and extend it to a global Minkowskian coordinate systems x^μ in $\mathcal{M}$. The holonomic components of the metric are constant and equal to the anholonomic components and eq. (1.1) shows that the components of the tetrad 4-vectors form a matrix of the orthochronous Lorenz group, namely

$$e^\mu_i = \Lambda^\mu{}_i. \tag{1.62}$$

This means that the holonomous and the anholonomous bases in the tangent spaces are related by Lorentz transformations. In the following we can replace the greek indices by latin indices.

A general element $s \in \mathcal{S}$ can be labelled by the 4-vector x and the Lorentz matrix Λ and it can be identified with the element (x, Λ) of the orthochronous

Poincaré group $\mathcal{P}$, with the usual multiplication law

$$(x, \Lambda)(x', \Lambda') = (x + \Lambda x', \Lambda\Lambda'). \tag{1.63}$$

The distinguished point $\hat{s}$ corresponds to the unit element $(0, 1)$.

The structural Lorentz group $\mathcal{L}$ is a subgroup of $\mathcal{P}$ and its action on $\mathcal{S} = \mathcal{P}$ is a *right translation*

$$(x, \Lambda) \to (x, \Lambda)(0, \Lambda') = (x, \Lambda\Lambda'). \tag{1.64}$$

It is clear that, in the particular case we are considering, the whole group $\mathcal{P}$ acts on $\mathcal{S}$ on the right and every element $s \in \mathcal{S}$ can be written in an unique way as $s = \hat{s}h$ with $h \in \mathcal{P}$. It follows that $\mathcal{S}$ is diffeomorphic to $\mathcal{P}$. It is important to remark, however, that this diffeomorphism depends on the choice of $\hat{s}$ and that the group $\mathcal{P}$ has a distinguished element, the unit, while no *a priori* privileged element is present in $\mathcal{S}$ (see Section 2.2). The infinitesimal right translations are generated by the vector fields A_α, which form a basis of the Poincaré Lie algebra.

One can also consider the *left translations* $s = \hat{s}h \to \hat{s}h'h$ that can be interpreted as changes $\hat{s} \to \hat{s}h'$ of the distinguished element $\hat{s}$. We indicate by A^L_α the generators of the left translations. They are vector fields on the group, but we can also interpret them as vector field on $\mathcal{S}$, though this interpretation depends on the choice of $\hat{s}$.

They commute with A_α and satisfy the commutation relations

$$[A^L_\alpha, A^L_\beta] = -\hat{F}^\gamma_{\alpha\beta} A^L_\gamma, \tag{1.65}$$

(note the minus sign). They can be written in the form

$$A^L_\alpha(s) = D^\beta{}_\alpha(h^{-1}) A_\beta(s), \qquad s = \hat{s}h, \qquad h \in \mathcal{P}, \tag{1.66}$$

where $D(h)$ is the adjoint representation of $\mathcal{P}$, which has the properties

$$A_\alpha D^\beta{}_\gamma(h) = D^\beta{}_\delta(h) \hat{F}^\delta_{\alpha\gamma}, \qquad A^L_\alpha D^\beta{}_\gamma(h) = \hat{F}^\beta_{\alpha\delta} D^\delta{}_\gamma(h). \tag{1.67}$$

We have the explicit formulas

$$\begin{gathered} D^i{}_k(x, \Lambda) = \Lambda^i{}_k, \qquad D^{[ik]}{}_{[jl]}(x, \Lambda) = \Lambda^i{}_j \Lambda^k{}_l - \Lambda^k{}_j \Lambda^i{}_l, \\ D^{[ik]}{}_j(x, \Lambda) = D^j{}_{[ik]}(0, \Lambda) = 0, \qquad D^j{}_{[ik]}(x, 1) = x_i \delta^j_k - x_k \delta^j_i. \end{gathered} \tag{1.68}$$

From the definition of left translation or also from eqs. (1.9), (1.31) and (1.66), one obtains

$$A^L_{[ik]} = g_{kj}\Lambda^j{}_l \frac{\partial}{\partial \Lambda^i{}_l} - g_{ij}\Lambda^j{}_l \frac{\partial}{\partial \Lambda^k{}_l} + x_k \frac{\partial}{\partial x^i} - x_i \frac{\partial}{\partial x^k}, \qquad A^L_i = \frac{\partial}{\partial x^i}. \quad (1.69)$$

In a flat spacetime one can give a clear definition of the *energy-momentum* four-vector p_i and the *relativistic angular momentum* tensor $p_{[ik]}$ of a system. A more general situation is discussed in Chapter 8. In the framework of analytical mechanics and of quantum theory, the quantities p_i and $p_{[ik]}$ are, respectively, the generators of the infinitesimal active translations and Lorentz transformations. Note that the energy, namely the Hamiltonian, is given by $H = p^0 = -p_0$ and it is the generator of the passive time translations. We have seen that the *passive transformations*, namely the transformations of the frame $\hat{s}$, are generated by the vector fields A^L_α and the corresponding *active transformations* are generated by the vector fields $-A^L_\alpha$.

Following the conventions of Section 1.7, it is natural to use also for the quantities p_i and $p_{[ik]}$ the compact notation p_α, and to call them the components of the *10-momentum*. By replacing $\mathcal{L}$ by $\mathcal{L} \times \mathcal{G}$ and $\mathcal{P}$ by $\mathcal{P} \times \mathcal{G}$, one can treat in a similar way the case in which gauge fields are present, but have a vanishing field strength. In this case one can define the *(10 + n)-momentum* of a system and for $\alpha = a = 10, \ldots, 9+n$, the quantities p_a are interpreted as the *charges* corresponding to the infinitesimal transformations of $\mathcal{G}$ and in particular $p_\bullet$ is the electric charge.

The quantites p_α, and the fields A^L_α, depend on the choice of the frame $\hat{s}$ in the same way. Since the fields A_α do not depend on $\hat{s}$, from eq. (1.66) we obtain the transformation formulas

$$\hat{s} \to \hat{s}h, \qquad A^L_\alpha \to D^\beta{}_\alpha(h) A^L_\beta, \qquad p_\alpha \to D^\beta{}_\alpha(h) p_\beta. \quad (1.70)$$

For infinitesimal transformations, we have

$$A_\gamma p_\alpha = \hat{F}^\beta_{\gamma\alpha} p_\beta. \quad (1.71)$$

This important formula will be discussed and generalized in Chapter 8. We have seen that the compact formalism can be extended to dynamical quantities.

If we use the explicit form of the adjoint representation, we obtain the usual Lorentz transformation properties of p_i and $p_{[ik]}$ and in the case of spacetime translations we have

$$\hat{s} \to \hat{s}(x, 1), \qquad p_i \to p_i, \qquad p_{[ik]} \to p_{[ik]} + x_i p_k - x_k p_i. \quad (1.72)$$

We also obtain the transformation formula for the charges p_a under a non-commutative internal symmetry group.

If we consider a single spinless point particle situated at the origin of the frame $\hat{s}(x, 1)$, $p_{[ik]}$ vanishes in this frame and in the frame $\hat{s}$ we have

$$p_{[ik]} = x_k p_i - x_i p_k, \tag{1.73}$$

If, in agreement with the conventions discussed in Section 1.7, we define the 3 dimensional vectors

$$\mathbf{p} = (p_1, p_2, p_3), \quad \mathbf{p}' = (p_{[32]}, p_{[13]}, p_{21]}), \quad \mathbf{p}'' = (p_{[01]}, p_{[02]}, p_{[03]}), \tag{1.74}$$

$$\mathbf{x} = (x_1, x_2, x_3), \tag{1.75}$$

we can write

$$\mathbf{p}' = \mathbf{x} \times \mathbf{p}, \quad \mathbf{p}'' = x^0 \mathbf{p} - p^0 \mathbf{x} \tag{1.76}$$

in agreement with the known elementary formulas.

A similar treatment can be given for a spacetime with constant curvature, namely a *de Sitter* or an *anti-de Sitter* spacetime. In this case the Poincaré group has to be replaced by the de Sitter or an anti-de Sitter group and the vector fields A_i do not commute, but we have

$$[A_i, A_k] = 2^{-1} \hat{F}_{ik}^{[jl]} A_{[il]}, \tag{1.77}$$

$$\hat{F}_{ik}^{[jl]} = -\rho(\delta_i^j \delta_k^l - \delta_k^j \delta_i^l), \tag{1.78}$$

where the constant ρ is connected with the scalar obtained by contraction of the Riemann curvature tensor by the relation

$$R = R^i_{jik} g^{jk} = -F_{ik}^{[ik]} = 12\rho. \tag{1.79}$$

It is positive for a de Sitter spacetime and negative for the anti-de Sitter spacetime. For the Minkowski space, we have $\rho = 0$.

The eqs. (1.12), (1.36) and (1.78) are equivalent to the formula

$$d\omega^i = -g_{jk} \omega^{[ij]} \wedge \omega^k, \qquad d\omega^{[ik]} = -g_{jl} \omega^{[ij]} \wedge \omega^{[lk]} + \rho \omega^i \wedge \omega^k. \tag{1.80}$$

Chapter 2

The general space $\mathcal{S}$ of the (extended) inertial local frames

2.1 The basic geometric and topological stucture of the space $\mathcal{S}$

In Chapter 1 the bundle $\mathcal{S}$ of Lorentz frames has been introduced as a useful auxiliary tool, while the spacetime $\mathcal{M}$ was considered as the basic geometric concept of physics. In the following, a theory based on a geometry of this kind is called a normal theory or, more exactly, a theory with a *normal geometry*. However, there are several arguments that suggest a change of perspective, namely that $\mathcal{S}$ has a more direct physical interpretation, while $\mathcal{M}$ should be considered as a mathematical construction, possibly justified only under some assumptions, which might have an approximate character.

In the presentation of these ideas we shall follow essentially refs. [3–5]. Similar points of views, with various motivations, have been presented by various authors [49–54]. Some of them are discussed with some detail in the follwing Sections.

A first simple argument in favour of the new point of view is that many physical observables have a vector or tensor nature and take a definite value only when a local frame, namely a point of $\mathcal{S}$ is given. A point $x \in \mathcal{M}$ does not convey enough information. Other arguments for choosing $\mathcal{S}$ as the geometric arena of physical theories will be given in the Sections 2.2 and 2.4.

We propose to drop the assumption that $\mathcal{S}$ is a principal fiber bundle with a connection and to endow it with a simpler structure suggested by the treatment given in Section 1.7 and justified by the operational discussion of Section 2.2. Further assumptions will be added when necessary.

More precisely, we describe the geometric background of a physical theory (including the gauge fields) by means of the $(10 + n)$-dimensional differentiable manifold $\mathcal{S}_n$ and the $10 + n$ differentiable vector fields A_α linearly independent at all the points $s \in \mathcal{S}_n$. The physical meaning of these fields given in Section 1.7 is only a useful suggestion. In Section 6.1 a slightly modified interpretation is suggested.

As we have remarked in Section 1.7, the vector fields A_α permit us to identify all the tangent spaces $T_s\mathcal{S}_n$ with a single $(10+n)$-dimensional vector space $\mathcal{T}_n$, namely they define on $\mathcal{S}_n$ a structure called *absolute parallelism* or *teleparallelism*. It can be described as a *trivialization* of the tangent bundle, namely a diffeomorphism between $T\mathcal{S}_n$ and $\mathcal{S}_n \times \mathcal{T}_n$

Adopting a more clear point of view, we define $\mathcal{T}_n$ as the vector space composed of all the vector fields of the form $A = b^\alpha A_\alpha$, where the coefficients b^α are constant. Note that all the vector fields on $\mathcal{S}_n$ can be written in the form $b^\alpha(s)A_\alpha(s)$ with variable $b^\alpha(s)$ and that the elements of $\mathcal{T}_n$ are vector fields of a particular kind, that we call *fundamental vector fields*. If $A \in \mathcal{T}_n$ is a fundamental vector field we have $A(s) \in T_s\mathcal{S}_n$ and in this way one establishes the isomorphism between $\mathcal{T}_n$ and $T_s\mathcal{S}_n$. The physical properties that charcterize the fundamental vector fields are discussed in Section 2.2.

We define the structure coefficients $F^\gamma_{\alpha\beta}$ by means of eq. (1.52) and the differential one-forms ω^β by means of eq. (1.53). They satisfy the equations (1.54) and (1.55), which have been discussed, in a particular context, in Section 1.7. Also the dynamical quantities which form the components of the $(10 + n)$-momentum (see Section 1.9), when they have an approximate meaning in the general formalism, are indicated by the compact notation p_α.

All the other properties of $\mathcal{S}_n$, and in particular its structure of principal fiber bundle with a connection, if it maintains an approximate validity, are shifted from the realm of geometry to the realm of dynamics. For instance, all the structure coefficients $F^\gamma_{\alpha\beta}$ should be considered as dynamical fields, namely their values should be determined by the field equations and the action principle. In the normal theories a dynamical role is recognized only to the coefficients F^γ_{ik}.

Also the equations (1.14) and (1.25), that determine the transformation properties of the fields with respect to the Lorentz and internal symmetry groups, should be considered as dynamical field equations, to be derived, as the other field equations which contain the spacetime derivatives, from the action principle.

If we consider an arbitrary constant positive definite $(10+n) \times (10+n)$ matrix $G_{\alpha\beta}$ and we consider its elements as the components of a metric tensor in the basis defined by the vector fields A_α, the manifold $\mathcal{S}_n$ acquires a structure of Riemannian manifold. One can easily see that the topology and the uniform structure defined by this metric do not depend on the choice of the matrix G. In particular, we can introduce without any ambiguity the concepts of completeness of the space $\mathcal{S}_n$ and of boundedness of a vector field. A vector field $b^\alpha A_\alpha$ is bounded if all its components b^α are bounded functions.

These concepts are very useful, because a theorem proven by Palais [55] assures that, if $\mathcal{S}_n$ is a complete Riemannian manifold, every bounded differentiable vector field A generates a one-parameter group of diffeomorphisms of $\mathcal{S}_n$ onto itself, that we indicate by $\exp(\tau A)$, where τ is a real parameter. More in general, a Lie algebra of bounded vector fields generates a right action of the corresponding simply connected Lie group on $\mathcal{S}_n$. The physical meaning of the completeness property is discussed in Section 2.5.

2.2 The operational interpretation and the relativity principle

Important ideas about the geometric structure of the space $\mathcal{S}$ and the role it plays in physical theories follow from an operational analysis of the geometric concepts of physics given in refs. [1, 2]. The operational point of view has been discussed by P. W. Bridgman [56] and an accurate presentation, which has strongly influenced our considerations, is given by R. Giles in ref. [57].

Of course, nobody is obliged to adopt the operational point of view. Any methodological choice is valid as soon as it helps to put some order in the physical experience [58]. We shall see that the operational analysis suggests several very interesting physical ideas.

According to ref. [57], a physical theory is a mathematical theory with an operational interpretation of some (not necessarily all) of its concepts (terms and relations). It is important to remember that some mathematical concepts may have no direct operational interpretation. For instance, spinor and charged fields are not observable [59, 60], but they are very useful in the formulation of a field theory.

An operational interpretation is based on physical (laboratory) operations. What is relevant, however, is not the single, concrete, operation, but a set of prescriptions, called a *procedure*, clearly stated in a specific document, which describes exhaustively how the operations have to be performed. For instance, in the description of a procedure it is not allowed to point the finger at some physical object. When this point of view is is rigorously accepted, we speak of *strictly operational interpretation*.

In order to specify the spacetime conditions, namely where and when the operation is performed and which has to be the velocity and the orientation of the instruments, the procedure must refer to some pre-existent physical object, chosen by the experimenter in any single case, which determines a "reference frame". In order to avoid confusion with the mathematical concept of reference frame, we use the term *situation*. A procedure does not specify how the situation has to be chosen. As we shall discuss in Section 2.6, it may be difficult to separate the geometric meaning of a situation from other physical information it necessarily contains.

The simplest kinds of procedures are the *measurement procedures*, which give a numerical result, and the *transformation procedures* which have the aim of building a new situation starting from a pre-existent one. More complicated procedures will be discussed in Section 3.4. It is convenient to call a *measurement* a class of equivalent measurement procedures that give (statistically) the same results in all the situations and, similarly, to call a *transformation* a class of equivalent transformation procedures that act (statistically) in the same way on the situations (a more precise definition is given in [2]).

One can define in a natural way the *composition of two transformations*, which is again a transformation, and of a transformation and a measurement, which is a new measurement. In agreement with our previous conventions, we write on the right the transformation performed later. In other words, the transformations form a semigroup acting on the right on the situations and on the left on the measurements. Other algebraic properties of the spaces of measurements and transformations are discussed in ref. [2].

It is important to remark that there is no possible strictly operational prescription for choosing a situation, unless a preceding situation is available. This means that the situations have no strict operational interpretation. The transformations have an operational interpretation and it is proposed in ref. [2] that the geometric concepts of physics should be defined in terms of transformations.

We see that the operational point of view leads to a *relational geometry*, in which the important concepts are not the frames (or the events), but the relations between them.

In the classical formalism we are considering, the manifold $\mathcal{S}$ is a model for the space of all the situations. Some problems raised by this definition are discussed in Section 2.6. We are considering the case $n = 0$, since the extended manifold requires a more delicate discussion.

Since the elements of $\mathcal{S}$ have no strict operational interpretation, the physical laws cannot privilege, and not even single out, any of them. It follows that all the points of $\mathcal{S}$, namely all the local inertial frames, have to be treated, *a priori* in the same way. This is a statement of the *relativity principle*, that appears as a consequence of the requirement that a physical theory must have a strict operational interpretation.

If one accepts this requirement, the relativity principle has to be considered as a part of the very definition of physics. Statements that privilege a particular local inertial frame do not belong to physics, but, possibly, to other sciences [23].

It is necessary to specify that only *a priori* equivalence of the inertial local frames is required. After the measurement of a field, the frames in which it takes a certain value may be privileged. This means that if one finds a seeming violation of the relativity principle, one has to find a field responsible for it. Of course, this field must have its own dynamics.

Originally, the relativity principle was restricted to pairs of frames which have different velocities. The formulation given above extends the principle to pairs of frames with different location in spacetime and different orientation in space. This general interpretation agrees with the ideas discussed in Section 2.4.

In the present notes we deal mainly with classical field theories, modelled on the Maxwell's and Einstein's theories, with the spacetime $\mathcal{M}$ replaced by $\mathcal{S}$. In particular, we assume that the measurement and the transformation procedures do not affect the state of the system. Some remarks about the limitations of this approach are given in Section 2.6. Then we can define a state of the system (including its time evolution, as in the Heisenberg picture of quantum mechanics), as a given solution of the field equations.

A measurement defines a scalar fields on $\mathcal{S}$, the value of the field at the point s being the outcome of the measurement performed in the local frame s. We are not assuming that all the scalar fields represent operationally defined measurements, but it seems natural to assume that the structure coefficients $F^{\gamma}_{\alpha\beta}$ are measurable fields.

A transformation induces a mapping of $\mathcal{S}$ into itself, that is assumed to be differentiable. We also assume the existence of one-parameter semigroups of transformations corresponding to mappings of the kind $s \rightarrow s\exp(\tau B)$ with $\tau \geq 0$. The vector field B describes an *infinitesimal transformation*.

Our main assumption is that the vector fields that describe infinitesimal transformations generate the 10-dimensional linear subspace $\mathcal{T}$ of the much larger space of all the vector fields in $\mathcal{S}$. The elements of $\mathcal{T}$ are called *fundamental vector fields* and define the absolute parallelism of $\mathcal{S}$, as it is explained in Section 2.1.

Only the vector fields belonging to a subset $\mathcal{T}^+ \subset \mathcal{T}$ generate a semigroup of transformations as we discuss in Section 3.1. The vector fields A_α form a basis of $\mathcal{T}$, but do not necessarily belong to $\mathcal{T}^+$. Of course, one can introduce a (global) change of basis, but a local (gauge) change of basis, different in the various tangent spaces $T_s\mathcal{S}$, is not admitted by our operational interpretation (see Section 3.8).

In order to justify our assumptions, we have to explain why a semigroup of diffeomorphisms of the kind

$$\exp(\tau b^\alpha(s) A_\alpha(s)), \tag{2.1}$$

where $b^\alpha(s)$ are suitably chosen nonconstant measurable scalar fields, cannot describe semigroup of transformations. Otherwise, we could define infinitesimal transformations represented by the vector field $b^\alpha A_\alpha$ not belonging to $\mathcal{T}$ and the absolute parallelism could not be defined in an unique way.

One may try to consider the transformation (2.1) as the composition of many transformations corresponding to small values of τ; in every step one measures the values of $b^\alpha(s)$ and then one performs the transformation as if these quantities were constant. One must remark, however, that it takes some minimum time to measure the values of the scalar fields $b^\alpha(s)$ and therefore the steps cannot be too small. As a consequence, we are authorized to assume that in the limit $\tau \rightarrow 0$ the mapping (2.1) can represent a transformation only if the fields $b^\alpha(s)$ are numerical constants known in advance.

Note that we had to assume that **it takes some minimum time to perform a measurement**. The generalization of this important principle to all the physical operations is discussed in Sections 2.4 and 3.1.

2.3 The spacetime coincidence

In a footnote of his fundamental article on General Relativity [61] Einstein wrote: "We assume the possibility of verifying simultaneity for events immediately proximate in space, or, to speak more precisely, for immediate proximity or coincidence in spacetime, without giving a definition of this fundamental concept".

In order to give a formal interpretation to this sentence, we consider a set $\mathcal{O}$ of "objects" and we assume that one can define operationally in a unique objective way an equivalence relation between them called *spacetime coincidence*. The equivalence classes are the elements of the spacetime $\mathcal{M}$, namely the *events*.

The elements of $\mathcal{O}$ are characterized by geometric properties and possibily also by dynamical properties, like energy, momentum, mass and so on. The interplay of geometric and dynamical quantities in theories with a modified spacetime has been discussed by several authors [62, 63] and finds perhaps its origin in refs. [64, 65].

If we disregard provisionally the dynamical properties, it is natural to identify the set $\mathcal{O}$ with the manifold $\mathcal{S}_n$. If it has a structure of fibre bundle as we have assumed in Chapter 1, the fibers are the equivalence classes for the relation of spacetime coincidence and Einstein's assumption is satisfied.

The full structure of principal fibre bundle is not necessary for a spacetime interpretation of the theory. We may more simply assume that the equivalence classes are $(6+n)$-dimensional differentiable submanifolds of $\mathcal{S}_n$ and that, with a suitable choice of a basis in $\mathcal{T}_n$, the vector fields A_α with $\alpha = 4, \dots, 9+n$ are tangent to them. The Lie bracket of two of these fields is also tangent to the equivalence classes and we obtain the condition

$$F^i_{\alpha\beta} = 0, \qquad \alpha, \beta = 4, \dots, 9+n, \qquad i = 0, \dots, 3. \tag{2.2}$$

One may ask if the condition (2.2) is sufficient for a spacetime interpretation. It means that the subspaces of $T_s\mathcal{S}_n$ generated by the vectors $A_4(s), \dots, A_{9+n}(s)$ (we continue to call them "vertical" subspaces) form an *integrable distribution*, namely they satisfy the condition of the Frobenius theorem [25, 26] which assures the existence of a *foliation* of $\mathcal{S}$. This means that for every point $s \in \mathcal{S}$ there are connected $(6-n)$-dimensional submanifolds, called *integral manifolds* containing s and tangent to the subspaces that form the distribution. One of these submanifolds contains all the others and is called the *leaf* containing s.

The same problem can also be treated by means of the forms ω^α. The vertical subspaces can be defined by means of the differential system

$$\omega^i = 0. \tag{2.3}$$

These differential forms must vanish when restricted to an integral manifold and their exterior derivatives

$$d\omega^i = -2^{-1} F^i_{\alpha\beta} \omega^\alpha \wedge \omega^\beta \tag{2.4}$$

must have the same property. In this way we obtain again the condition (2.2), which is the integrability condition for the differential system (2.3).

We can consider the leaves as the equivalence classes of a relation of spacetime coincidence and consider them as the elements of a set $\mathcal{M}$. We indicate by π the projection of $\mathcal{S}$ on $\mathcal{M}$ and we say that a set $I \subset \mathcal{M}$ is open if its inverse image $\pi^{-1}(I)$ is open. With this definition $\mathcal{M}$ is a topological space, but the *Hausdorff separation axiom* is not necessarily satisfied.

A minor problem is that, at least if $\mathcal{S}$ is a bundle of frames, every equivalence class (fiber) has two connected components containing left-handed and right handed tetrads. As a consequence it is composed of two leaves and every point of the spacetime is obtained twice. Alternatively, one can say that one obtains a double covering of $\mathcal{M}$.

One can also try to define a structure of differentiable manifold on $\mathcal{M}$. In the proof of Frobenius' theorem one shows that in a neighborhood of every point of $s \in \mathcal{S}_n$ one can find a cubic coordinate system ξ^α with $\|\xi^\alpha\| < d$, in such a way that the surfaces (slices) defined by fixing the values of the four coordinates $\xi^0, \ldots, \xi^3$ are integral manifolds and therefore individuate a point of $\mathcal{M}$. From a local point of view we can consider the functions $\xi^0, \ldots, \xi^3$ as local spacetime coordinates and say that, as a consequence of eq. (2.2) the theory has a *local spacetime interpretation*.

From a global point of view, the functions $\xi^0, \ldots, \xi^3$ provide a local coordinate system of $\mathcal{M}$ only if different values of them correspond to different points of $\mathcal{M}$, namely if different slices belong to different leaves. In this case we say that the cubic coordinate system is regular. If near every point of $\mathcal{S}$ one can find a *regular cubic coordinate system*, one can consider $\mathcal{M}$ as a differentiable manifold and a *global spacetime interpretation* is established.

Unfortunately, often this regularity condition is not satisfied. It is useful to illustrate the situation by means of a simple example. We assume that the Poincaré group $\mathcal{P}$ acts on $\mathcal{S}$ on the right transitively, but not freely. The infinitesimal transformations are described by the vector fields A_α and the structure coefficients are the structure constants of the Poincaré Lie algebra, so that the condition (2.2) is satisfied. The manifold $\mathcal{S}$ is a homogeneous space and is described by the quotient $\mathcal{H}\backslash\mathcal{P}$ where $\mathcal{H}$ is the stabilizer of a given element $\hat{s} \in \mathcal{S}$, a closed subgroup of $\mathcal{P}$.

We assume that $\mathcal{H}$ is the Abelian discrete subgroup containing the elements

$$\exp\left(pA_{[01]} + (pa + qb)A_2\right), \tag{2.5}$$

where p and q are integers, a and b are real and $a^{-1}b$ is irrational. The numbers of the form $pa + qb$ are dense on the real line and one can find the sequences $\{p_r\}$ and $\{q_r\}$ with the properties

$$\lim_{r\to\infty} p_r = +\infty, \qquad \lim_{r\to\infty} (p_r a + q_r b) = 0. \tag{2.6}$$

We obtain

$$\lim_{r\to\infty} \hat{s}\exp\left(p_r A_{[01]}\right) = \lim_{r\to\infty} \hat{s}\exp\left(-(p_r a + q_r b)A_2\right) = \hat{s} \tag{2.7}$$

All the elements of this sequence belong to the same leaf, but not to the same slice, since $p_r a + q_r b \neq 0$. It follows that the regularity condition is not satisfied.

This example shows that there is no hope to assure the existence of a global space-time interpretation by means of local conditions on the structure coefficients. Since we are mainly interested in the study of local field equations, we shall not consider this problem any more.

The equivalence relation considered above can be called *primary local spacetime coincidence*, because it is defined directly by means of the fundamental vector fields A_α. In Chapter 7 we define a more general concept of *secondary local spacetime coincidence*, that also depends on the structure coefficients.

If $n > 0$, one can also define the weaker concept of *physical equivalence* of extended frames. We say that two extended frames are physically equivalent if they differ by a gauge transformation, namely they belong to the same fiber of $\mathcal{S}_n$ considered as a principal fiber bundle with base $\mathcal{S}$ and structural group $\mathcal{G}$. The whole treatment given above can be repeated in this different context and we find the necessary condition

$$F^\alpha_{ab} = 0, \qquad a, b = 10, \ldots, 9+n, \qquad \alpha = 0, \ldots, 9. \tag{2.8}$$

2.4 The equity principle, the minimum time principle and the fundamental length

Another argument in favour of the use of the space of frames $\mathcal{S}$ in the formulation of field theories has been pointed out by Lurçat [49, 50] in 1964. At that time a large part of the research in elementary particle physics was devoted to the relation between mass and spin described by "Regge trajectories" [66] and it was realized that spin is not an "unessential complication" and has a dynamical role.

It was suggested that energy-momentum and spin are equally important and should be treated on the same footing. As a consequence, a unified treatment of spacetime translations and rotations was desirable. This argument suggested a formalism in which quantum fields are defined on the Poincaré group, that, as we have shown in Section 1.9, can be identified with the bundle of the Lorentz frames of the Minkowski spacetime.

These ideas were suggested by strong interaction physics, but we apply them to the microscopic physics of space-time and gravitation. Also the ideas of string theory, one of the most popular candidates for the solution of the problems of quantum gravity, had their origin in the theory of strong interactions.

From the point of view of the space $\mathcal{S}$, vertical and horizontal vectors, which represent Lorentz transformations and (parallel) translations, should be treated on an equal footing, but this is not compatible with the structure of principal bundle. The treatment given in Chapter 1 shows that a considerable part of the geometric concepts presented there does not distinguish between vertical and horizontal vectors and we have employed some effort in order to put in evidence these aspect. This is particularly evident in the compact formalism of Section 1.7.

We propose the term *equity principle* for the idea that mass and spin, translations and rotations, **horizontal and vertical subspace should be treated on an equal footing**. There is no operational justification for this principle as, for instance, for the relativity principle (see Section 2.2). In fact, spacetime translations and Lorentz transformations have different operational interpretations.

Also the space and time translations have different operational definitions, but relativity theory treats them in a symmetric way, in the sense that the Lorentz group has an irreducible action on the four-vector space.

In Chapter 3 we implement the equity principle by introducing a group that acts irreducibly on the space $\mathcal{T}$. This means, in particular, that the vertical and the horizontal subspaces cannot be invariant.

The equity principle has mainly an heuristic value, namely it is useful for the construction of new theories, to be compared with experiments. It is known that after pruning a fruit tree, it grows stronger and it gives more and better fruits. A similar occurrence has been observed several times in the history of physics, the best known examples being the absolute time in relativity and the electron orbits in quantum atomic physics. It is possible that dropping the assumptions that spoil the symmetry between vertical and horizontal vectors can help in the treatment of the problems that affect the theories of spacetime and quantum gravity.

As it is discussed in Section 2.3, the vertical subspace is strictly related to the idea of spacetime coincidence. If the equity principle is accepted, the absolute character of the space-time coincidence, a fundamental assumption of General Relativity, has to be abandoned.

One should also remark that translations and rotations are described, respectively, by a length and by an adimensional parameter (angle or velocity). It follows that a symmetric treatment requires the introduction of a fundamental lenght ℓ, in the same way as the introduction of a fundamental velocity c is necessary for a symmetric treatment of space and time in special relativity. The introduction of a fundamental length in the classical theory of gravitation should help to eliminate the singularities that appear in the black hole and in the big bang solutions.

We have to discuss the relation between the fundamental length ℓ appearing in a (still hypothetic) modified classical theory of gravity and the Planck length

$$\ell_P = (\hbar G)^{1/2}, \tag{2.9}$$

which appears in all the attempts to quantize General Relativity. If $\ell \gg \ell_P$, some (not yet observed) effects of quantum gravity could be masked by the effects of ℓ. However, we think that one should find some reason why $\ell = \nu \ell_P$, where ν is a constant of the order of one.

One could consider the modified classical theory as an approximation of the quantum theory in a situation in which the effects of $\hbar$ can be disregarded, but the effects of ℓ_P are still present. For instance, if one considers particles with an extremely large energy-momentum dispersion, the spacetime uncertainty introduced by ℓ could be more important than the uncertainty introduced by the Heisenberg relations.

From a more formal point of view, one could remark that in a classical theory ℓ and G are two independent parameters and that in many cases a classical system can be quantized only if a relation between its parameters and $\hbar$ is satisfied [67, 68]. The simplest example is given by a classical spin system described by a phase space given by 2-dimensional sphere with a symplectic form proportional to the rotation invariant element of area and normalized in such a way that the total area is a. The classical system makes sense for any positive value of the constant a, which has the dimension of an action, but quantization is possible only if a is an integral multiple of $2\pi\hbar$. The relation $\ell = \nu \ell_P$ could have a similar origin. Further suggestions are given in Section 4.5.

In another scenario, the parameters ℓ, G and $\hbar$ are independent and the quantized theory has a perturbative expansion in the adimensional parameter $G\hbar\ell^{-2}$ (of course for $\ell > 0$), but a singularity appears when the parameter reaches a value of the order of one. We are not proposing a quantum theory of gravitation, but we only want to show that a classical theory of gravitation containing a fundamental length is not *a priori* wrong. With a bit of imagination, one could find in this way an explanation for the difficulties found in the quantization of a normal gravitational theory with $\ell = 0$.

We have assumed that G is a fundamental constant and that the classical theory contains another fundamental constant ℓ with the dimension of a length. A function of these constants has the dimension of an action and, as we have discussed above, it must have some connection with the fundamental constant $\hbar$ introduced by quantization. In this situation, we say that the classical theory is "prepared" for quantization.

However, several authors [69–71] have proposed theories with variable G. These theories have not been confirmed experimentally, but they have several interesting features and deserve a serious consideration. If G is a variable, and ℓ is the only fundamental constant (besides the velocity of light), the classical theory is not "prepared" for quantization unless the fundamental constant ℓ, or some power of it, has the dimension of an action. We shall discuss further this argument in Section 6.3.

The analogy between translations, rotations and Lorentz boosts implies the analogy between velocity, angular velocity and acceleration. Since in relativistic theories there is an upper bound to the velocity, the equity principle suggests the existence of upper bounds of the order of ℓ^{-1} to the angular velocity and the acceleration. A maximal acceleration has been suggested, with various motivations, by many authors [7, 14, 72–77]. A more detailed discussion is given in Section 3.1 and in Chapter 8.

The same concepts can be formulated in a different way. The existence of a maximal velocity means that it takes a minimum time to perform a space translation. The equity principle implies that it also takes a minimum time to perform rotations and boosts. We have already suggested in Section 2.2 that it takes a minimum time to perform measurements. It is natural to formulate a *minimum time principle* requiring that **it takes a minimum time to perform any physical operation.**

We assume that mathematical operations can be performed in an arbitrarily short time. In particular, we do not apply the minimum time principle to gauge transformations, usually considered as a purely mathematical change in the description of physical phenomena. An investigation of problems of this kind requires an interaction between physics and inforamtion theory, that we are not prepared to tackle.

2.5 Dynamical variables and symmetry properties

The dynamical variables of a classical field theory on the space $\mathcal{S}_n$ are the vector fields A_α and a set of scalar fields Ψ^U that we consider as the elements of a one-column matrix Ψ. The vector fields describe the geometry and the scalar fields represent "matter". If $n > 0$, the internal gauge fields are, by convention, considered as geometric fields.

It is not necessary to consider tensor fields in the manifold $\mathcal{S}_n$, because they are described in terms of scalar fields, namely their components with respect to the frame defined by the vector fields A_α. For the same reason the use of differential forms can be avoided, but it often permits more elegant and compact formulas.

According to the relativity principle discussed in Section 2.2, the field equations cannot privilege any local frame and must be invariant with respect to general transformations of the local coordinates of $\mathcal{S}_n$ or, in other words, they must be invariant under *diffeomorphisms* of $\mathcal{S}_n$.

This condition is automatically satisfied if we use the coordinate-free formalism of differential geometry. We assume that the field equations have a local character and that they consist of algebraic relations involving, at any point of $\mathcal{S}_n$, the scalar fields, the structure coefficients and their derivatives expressed by means of the differential operators A_α.

A solution determines, besides the values of the vector and scalar fields, the structure of $\mathcal{S}_n$ as a differentiable manifold, in particular its topology. We say that a solution is *complete* if the metric space $\mathcal{S}_n$ is complete (see Section 2.1). If $\mathcal{S}_n$ is not connected, we assume that its connected components do not describe different noncommunicating universes and that the values of the fields on one components already describe a solution completely. Then, one can assume that $\mathcal{S}_n$ is *connected.* For instance, one can consider only left-hande tetrads, disregarding the right-handed ones..

Given a complete solution, one can consider an open proper submanifold of $\mathcal{S}_n$, describing a new solution, that we may call a *subsolution.* If $\mathcal{S}_n$ is connected, this submanifold cannot be open and closed at the same time and therefore it cannot be complete. It follows that a complete solution cannot be a subsolution of a larger connected solution. Note that it certainly has a completion, but it is not necessarily a manifold. From the physical point of view, a complete connected solution is intended to give a description of the whole universe. If a noncomplete solution cannot be described as a subsolution of a complete solution, it means that it has singularities [33].

A solution is called *constant* if all the scalar fields Ψ^U and all the structure coefficients are constant. It follows from eq. (1.55) that the structure coefficients are the structure constants of a Lie algebra that generates a simply connected $(10+n)$-dimensional Lie group $\mathcal{P}_n$. If the solution is also connected and complete, this group acts transitively on $\mathcal{S}_n$, which is diffeomorphic to the *homogeneous space* $\mathcal{H}\backslash\mathcal{P}_n$, where $\mathcal{H}$ is a discrete subgroup of $\mathcal{P}_n$. A constant solution is often interpreted as a *vacuum state* of the theory.

An example of field equation is eq. (1.55), which is always valid. It is invariant with respect to all the changes of basis in the space $\mathcal{T}_n$, namely with respect to the group $GL(10+n, \mathbf{R})$ acting on the Greek indices. In general, the other field equations and the Lagrangian form that generates them have a different symmetry group $\mathcal{F}$ which transforms the fields according to the formulas

$$\Psi \to S(k)\Psi, \qquad A_\alpha \to (C^{-1})^\beta{}_\alpha(k)A_\beta, \qquad \omega^\alpha \to C^\alpha{}_\beta(k)\omega^\beta, \qquad (2.10)$$

where $k \in \mathcal{F}$ and $S(k)$, $C(k)$ are linear representations of $\mathcal{F}$. The representation C is real and one can consider it as acting on the vector space $\mathcal{T}_n$.

If we indicate by κ an element of the Lie algebra of $\mathcal{F}$, and by $S(\kappa)$ and $C(\kappa)$ the representations of the Lie algebra corresponding to the representations of the group, the infinitesimal transformations of the dynamical variables are given by

$$\delta\Psi = \zeta S(\kappa)\Psi, \qquad \delta A_\alpha = -\zeta C^\beta{}_\alpha(\kappa) A_\beta, \qquad \delta\omega^\alpha = \zeta C^\alpha{}_\beta(\kappa)\omega^\beta. \tag{2.11}$$

The group $\mathcal{F}$ may contain elements that do not act on $\mathcal{T}_n$, for instance the isotopic spin transformations and other transformations that act on the flavour indices. They form a closed invariant subgroup $\mathcal{K} \subset \mathcal{F}$ which is the kernel of the representation $C(k)$. The quotient group $\mathcal{F}^G = \mathcal{F}/\mathcal{K}$ is the *geometric symmetry group* and $C(k)$ can be considered as a faithful representation of it. In the simplest cases we have $\mathcal{F} = \mathcal{F}^G \times \mathcal{K}$.

In all the cases we shall consider $\mathcal{F}^G$ contains the Lorentz group $\mathcal{L}$ that does not act on the subspace $\mathcal{T}_I$ and possibly the internal gauge group $\mathcal{G}$ that does not act on the subspaces $\mathcal{T}_H$ and $\mathcal{T}_L$. The action of the Lorentz group on the subspaces $\mathcal{T}_H$ and $\mathcal{T}_L$ can be written in the form

$$A_i \to (\Lambda^{-1})^k{}_i A_k, \qquad A_{[ik]} \to (\Lambda^{-1})^j{}_i (\Lambda^{-1})^l{}_k A_{[jl]} \tag{2.12}$$

and the corresponding infinitesimal transformations are given by

$$\begin{aligned} \delta A_\alpha = -2^{-1}\zeta^{[ik]}\hat{F}^\beta_{[ik]\alpha} A_\beta, \qquad \delta\omega^\alpha &= 2^{-1}\zeta^{[ik]}\hat{F}^\alpha_{[ik]\beta}\omega^\beta, \\ \delta\Psi &= 2^{-1}\zeta^{[ik]}\Sigma_{[ik]}\Psi. \end{aligned} \tag{2.13}$$

We see that the subspaces $\mathcal{T}_H$, $\mathcal{T}_L$ and $\mathcal{T}_I$ are invariant and the representation C is reducible. A possible mathematical formulation of the equity principle (see Section 2.4) is to introduce a larger group $\mathcal{F}^G$ and to require that the representation C, restricted to $\mathcal{T}_H \oplus \mathcal{T}_L$ (still assumed to be invariant) is irreducible, so that there is no invariant definition of the horizontal and vertical subspaces. We shall treat this problem in Chapter 7. One may invent more complicated symmetry group, but we do not see any physical motivation.

A classification of the possible groups $\mathcal{F}^G$ containing the Lorentz group has been given in ref. [6], but it was realized later that the number of interesting solutions is considerably reduced by requiring the existence of an invariant cone in the tangent spaces of $\mathcal{S}$, as it is discussed in Chapter 3.

As we have already remarked, a different kind of symmetry of the field equations is given by the diffeomorphisms of the manifold $\mathcal{S}_n$. An infinitesimal diffeomorphism is generated by a vector field B and its action on the fields is described by the *Lie derivative* $L(B)$. The action on scalar and vector fields and on the differential forms is given by [25, 26]

$$L(B)\Psi = B\Psi, \qquad L(B)A = [B, A], \qquad L(B)\omega = i(B)d\omega + di(B)\omega. \quad (2.14)$$

A diffeomorphism acting on the manifold $\mathcal{S}_n$ and on the vector and scalar fields transforms a solution into a physically equivalent one, namely it is a *gauge transformation*. The usual gauge transformations of a normal theory are described by the right action on the fibers of an element of the structural group that can depend on the fiber. In the general theories we are considering all the diffeomorphisms have to be considered as gauge transformations.

The most general infinitesimal symmetry transformation is described by a pair (B, κ), where κ belongs to the Lie algebra of $\mathcal{F}$ and is given by

$$\delta\Psi = \zeta(B\Psi + S(\kappa)\Psi), \qquad \delta A_\alpha = \zeta([B, A_\alpha] - C^\beta{}_\alpha(\kappa)A_\beta). \quad (2.15)$$

It is important to avoid any confusion between the symmetry group of the field equations and the symmetry group of a given solution. A pair (B, κ) describes a symmetry of a solution if $\delta\Psi = 0$ and $\delta A_\alpha = 0$. If we require only the second equality, we obtain the symmetry group of the geometric aspects of the solution. This is the symmetry group of a theory dealing with some matter fields in a fixed geometric background. Some classical and quantum theories of this kind are treated in ref. [17] and in Chapter 11.

It is interesting to consider solutions in which $\mathcal{S}_n$ is a bundle of frames. An element of the structural group defines a diffeomorphism of $\mathcal{S}_n$, which in general affects the vector fields A_α. Only if one can compensate this effect by means of a transformation belonging to $\mathcal{F}^G$, one obtains an element of the symmetry group of the geometry of a solution. For instance, an infinitesimal Lorentz transformation of the kind (2.13) combined with the infinitesimal diffeomorphism generated by $B = 1/2\zeta^{[ik]}A_{[ik]}$ gives $\delta A_\alpha = 0$. A similar argument holds for the space inversion.

2.6 Critical remarks

It is evident that there is a serious gap between the operational approach discussed in refs. [1, 2] and summarized, with several simplifications, in Section 2.2 and the geometric structure described in Section 2.1. We do not even try to fill this gap, but we think that it is useful to point out some of its aspects.

It looks more and more probable that understanding the very small scale features of the geometric concepts requires a paradigm shift [78] involving several aspects of science. We like to think that the considerations given in the present notes give a small contribution in this direction, but much more work is necessary, expecially concerning the quantum aspects of the problem. The point of view of ref. [2] is more radical, but it is very difficult to formulate in that way a sufficiently rich amount of physical knowledge.

A first simplification present in our approach is to disregard the casual features present in every physical operation, in particular the experimental errors. They, far from being a nuisance, provide an operational justification for the topological concepts of the theoretical models, as it is explained in ref. [57]. The connection between experimental uncertainties and the topological concepts has been pointed out by Poincaré one century ago [79].

By allowing the presence of casual choices in the description of a procedure, one can endowe the varions spaces of procedures with a structure of convex set, similar to the structure of the space of the mixed states in quantum mechanics. We do not use in the following this very useful structure.

A second problem is given by the doubts raised by the interpretation of the manifold $\mathcal{S}$ as the collection of all the possible local inertial frames. The local frames, called more exactly "situations" in ref. [2], are defined by material objects that inavoidably interact with each other and with the physical objects under investigation. It is clearly impossible to realize all of them. One could define $\mathcal{S}$ as the collection of all the "potential" local frames, but it is difficult to understand what this means.

It may be useful to turn to a fiction, namely assume that the local frames are defined by a very "thin" kind of idealized matter that interacts appreciably only with the measuring instruments, in order to transmit to them the geometric informations. If this fiction represents a good approximation, we say that the (classical or quantum) theory has a *classical geometry*. However, it is inavoidable to take into account:

- the gravitational field generated by the objects that define the frames;
- their quantum properties that do not permit an exact determination of their positions and of their velocities.

The gravitational field can be disregarded if the mass of the object that defines the reference frames is sufficiently small and the quantum effects are not important if this mass is sufficiently large. Only in physical situations in which one can choose a mass that satisfies both these conditions one can adopt a classical geometry. Otherwise, one enters the realm of quantum geometry, namely of quantum gravity. Quantum frames have been discussed in refs. [18, 80–82].

A third problem concerns the concept of state. In atomic or particle physics a state is defined operationally by a *preparation procedures*, for instance a suitable instrument produces a beam of atoms with given quantum numbers. Alternatively, one can take a large set of atoms, measure a complete set of observables and choose the atoms that have given the required outcomes. It is clear that these methods do not work when one is investigating a large part of the universe. Other problems arise if one consider with more detail the interpretation of quantum mechanics [83].

Actually, one can formulate a physical theory only in terms of observables (measurement procedures) and relations between them. This is the point of view adopted in refs. [1, 2]. Some functions of the concept of state are transferred to the concept of situation. Instead of measuring the observables in various different states, one measures them in various randomly chosen situations and performs a statistical analysis of the outcomes. This is possible because we live in a very large universe and we can choose an arbitrarily large set of situations separated is space and in time. Then, if one likes, one can introduce the states as functionals defined on the space of measurements.

This merging of the concept of state and the concept of local frame in the concept of situation, is, in our opinion, a probable feature of a future theory of physical geometry. The merging of different operationally defined concepts when one enlarges their range of application is a general phenomenon analyzed in ref. [56]. The best known example is the merging of the operational definitions of energy and frequency when one applies them to microscopic physics.

Since we are dealing with classical field theories, we can freely define a state as a solutioon of the field equations, as we have done in Section 2.5. The outcomes of all the observables measured in arbitrary local frames are given by the values taken by the fields. This is possible in a classical theory, since all the observables are compatible and their measurement does not affect the state of the system.

We think that it is correct to develop, as we are doing in the present notes, the drastically simplified classical scheme based on the manifold $\mathcal{S}$, even if it can be criticized from a methodological point of view, as soon as it can be used to formulate useful physical ideas [58]. A similar and perhaps more severe criticism applies to the classical field theories based on the spacetime manifold $\mathcal{M}$.

Chapter 3

Feasibility of infinitesimal transformations and geometric symmetry groups

3.1 A wedge $\mathcal{T}^+$ in the vector space $\mathcal{T}$

In Section 2.2 we have introduced the n-dimensional vector space $\mathcal{T}$ generated by its subset $\mathcal{T}^+ \subset \mathcal{T}$ which contains all the vector fields that describe *feasible infinitesimal transformations*, namely semigroups of operationally defined transformations. Note that not all the elements of $\mathcal{T}$ belong to $\mathcal{T}^+$. For instance, the vector field $-A_0$ generates time translations in the past, which cannot be realized. In other terms, one cannot build a situation (local frame) in the past. The properties of $\mathcal{T}^+$ have been discussed in refs. [7, 11, 12, 14].

It is clear that $\mathcal{T}^+$ is dilatation invariant and we also assume that it is closed. If $A, B \in \mathcal{T}^+$, we have [84]

$$\lim_{k\to\infty} \Big(\exp(k^{-1}\tau A)\exp(k^{-1}\tau B)\Big)^k = \exp(\tau(A+B)) \tag{3.1}$$

and we see that $A + B$ generates a semigroup of feasible transformations, namely $A + B \in \mathcal{T}^+$. It follows that $\mathcal{T}^+$, besides being dilatation invariant, is convex, namely it is a *wedge* [7, 11]. Since it generates the whole vector space $\mathcal{T}$, it has a nonempty interior.

A *cone* is a wedge that does not contain straight lines. If $\mathcal{T}^+$ is not a cone, the linear subspace $\mathcal{T}^+ \cap -\mathcal{T}^+$, that contains the *reversible infinitesimal* *transformations*, has positive dimension. We consider two vector fields A, B belonging to this subspace. Since $A, B, -A, -B \in \mathcal{T}^+$, from the formula [84]

$$\begin{aligned}\lim_{k\to\infty}\left(\exp(k^{-1}\tau A)\exp(k^{-1}\tau B)\exp(-k^{-1}\tau A)\exp(-k^{-1}\tau B)\right)^{k^2}\\ = \exp(\tau^2[AB]),\end{aligned} \tag{3.2}$$

we see that the commutators $[AB]$ and $[BA]$ generate semigroups of feasible transformations. It follows that $[AB] \in \mathcal{T}^+ \cap -\mathcal{T}^+$ and this subspace defines an involutive distribution of subspaces in the tangent spaces of $\mathcal{S}$.

It is convenient to write a generic vector of $\mathcal{T}^+$ in the form

$$B = b^\alpha A_\alpha = b^i A_i + 2^{-1} b^{[ik]} A_{[ik]}. \tag{3.3}$$

We also introduce the 3-dimensional vectors

$$\mathbf{b} = (b^1, b^2, b^3), \qquad \mathbf{b}' = (b^{[32]}, b^{[13]}, b^{[21]}), \qquad \mathbf{b}'' = (b^{[01]}, b^{[02]}, b^{[03]}). \tag{3.4}$$

In the normal theories all the Lorentz transformation and the spacetime translations belonging to the closed future cone are feasible, namely $\mathcal{T}^+$ is a wedge defined by the inequality

$$b^0 \geq \|\mathbf{b}\|, \tag{3.5}$$

while $\mathbf{b}'$ and $\mathbf{b}''$ are arbitrary. It follows that

$$\mathcal{T}^+ \cap -\mathcal{T}^+ = \mathcal{T}_V. \tag{3.6}$$

Note that eq. (2.2), that permits a local spacetime interpretation, is a consequence of the structure of $\mathcal{T}^+$.

One may say that in a normal theory it takes some time to perform a space translation, but it takes no time to perform a rotation or a Lorentz boost. Of course, it takes no time to perform internal gauge transformations.

3.2 The Lorentz invariant cone $\mathcal{T}^+$

Assuming $m = 0$, namely disregarding the internal gauge transformations, if we want to satisfy the equity and the minimum time principles (see Section 2.4), a nonvanishing element of $\mathcal{T}^+$ must have $b^0 > 0$, since all the physical operations "take some time". It follows that $\mathcal{T}^+$ is a **closed cone with a**

nonempty interior and we call it the *feasibility cone*. We say that a theory of this kind has a *modified geometry*. In the presentation of this argument we follow refs. [7, 11, 14].

In order to determine the structure of $\mathcal{T}^+$, we have to assume some symmetry property and a natural requirement is that, as in the normal theories, it is symmetric under the proper orthochronous Lorentz group $\mathcal{L} = SO^{\uparrow}(1,3)$. More precisely, we write an element of $\mathcal{T}^+$ in the form (3.3) and we require the invariance under the Lorentz transformations

$$b^i \to \Lambda^i{}_k b^k, \qquad b^{[ik]} \to \Lambda^i{}_j \Lambda^k{}_l b^{[jl]}. \tag{3.7}$$

If $\mathcal{T}^+$ contains an element with coordinates $(b^0, \mathbf{b}, \mathbf{b}', \mathbf{b}'')$, it contains also the element with co-ordinates $(b^0, \mathbf{0}, \mathbf{0}, \mathbf{0})$, which belongs to the convex hull of a finite set of points obtained from the given point by means of suitable rotations. The dilatation invariant set of the possible values of b^0 cannot be reduced to 0, because $\mathcal{T}^+$ generates $\mathcal{T}$ and cannot be the whole real line because $\mathcal{T}^+$ is a cone. It follows that it is the half line $b^0 \leq 0$ or the half line $b^0 \geq 0$. For physical reasons, we choose the second possibility. In order to avoid that, after a Lorentz transformation, b^0 becomes negative, we have to assume that $\mathcal{T}^+$ is contained in the wedge defined by eq. (3.5).

We consider an arbitrary element of the interior of $\mathcal{T}^+$ and we simplify its coordinates by means of a suitable Lorentz transformation. Since the four-vector b is timelike, we can obtain $\mathbf{b} = \mathbf{0}$. Then, by means of a rotation, we can cancel the third components of $\mathbf{b}'$ and $\mathbf{b}''$. In conclusion, we obtain

$$b^0 > 0, \qquad b^1 = b^2 = b^3 = b^{[21]} = b^{[03]} = 0. \tag{3.8}$$

We indicate by $b^\alpha(\zeta)$ the coordinates obtained from b^α by means of a Lorentz boost with rapidity ζ along the third space axis and we define the quantities

$$b^\alpha_\pm = 2 \lim_{\zeta \to \pm\infty} \exp(-|\zeta|) b^\alpha(\zeta). \tag{3.9}$$

Since $\mathcal{T}^+$ is Lorentz and dilatation invariant and it is also closed, it contains the points with co-ordinates $b^\alpha_\pm$. One can easily see that the only nonvanishing limits are

$$\begin{aligned} b^0_\pm &= b^0, \quad b^{[01]}_\pm = b^{[01]} \pm b^{[31]}, \quad b^{[32]}_\pm = b^{[32]} \pm b^{[02]}, \\ b^3_\pm &= \pm b^0, \quad b^{[02]}_\pm = b^{[02]} \pm b^{[32]}, \quad b^{[13]}_\pm = b^{[13]} \pm b^{[10]} \end{aligned} \tag{3.10}$$

and it follows that

$$b^\alpha = 2^{-1}(b_+^\alpha + b_-^\alpha). \tag{3.11}$$

We have shown that the element considered is the sum of two elements of $\mathcal{T}^+$ that satisfy the conditions

$$\begin{aligned} \|\mathbf{b}\| = b^0 > 0, \qquad \|\mathbf{b}'\| = \|\mathbf{b}''\| = ab^0, \\ \mathbf{b}' \cdot \mathbf{b}'' = 0, \qquad \mathbf{b}' \times \mathbf{b}'' = a^2 b^0 \mathbf{b}. \end{aligned} \tag{3.12}$$

These conditions are equivalent to the Lorentz invariant conditions

$$\begin{aligned} b^0 > 0, \qquad b^k b_k = 0, \qquad b^{[ik]} b_{[ik]} = 0, \\ \epsilon_{ikjl} b^{ik} b^{jl} = 0, \qquad b^{ji} b_{jk} = a^2 b^i b_k \qquad a \geq 0 \end{aligned} \tag{3.13}$$

and, for any given $a \geq 0$, define a set $\mathcal{T}_a \subset \mathcal{T}$ invariant with respect to the proper orthochronous Lorentz group $\mathcal{L}$. It follows that the decomposition found above in a special case is also possible for an arbitrary element of the interior of $\mathcal{T}^+$.

One can easily see that $\mathcal{L}$ acts transitively on the sets $\mathcal{T}_a$, which are either contained in $\mathcal{T}^+$ or do not intersect it. The first possibility is realized if a belongs to a closed convex set containing the point $a = 0$ and bounded, otherwise for fixed b^0 arbitrarily large values of $\mathbf{b}'$ and $\mathbf{b}''$ would be permitted and $\mathcal{T}^+$ would not be a cone. In other words, we must have $0 \leq a \leq \ell^{-1}$, where ℓ is a positive fundamental length. One can easily see that if this inequality is satisfied, every element of $\mathcal{T}_a$ can be written as the sum of two elements of $\mathcal{T}_{1/\ell}$.

In conclusion, we have proven that all the elements of the interior of $\mathcal{T}^+$ can be written as the sum of two elements of $\mathcal{T}_a$ with $a \leq 1/\ell$ or of four elements of $\mathcal{T}_{1/\ell}$. Since $\mathcal{T}^+$ is the closure of its interior, it is easy to show that all its elements have the same decomposition. Note that, if $B \in \mathcal{T}_{1/\ell}$, $(b^0)^{-1}\mathbf{b}$, $\ell(b^0)^{-1}\mathbf{b}'$ and $\ell(b^0)^{-1}\mathbf{b}''$ form a left-handed triad of normalized orthogonal vectors.

From the decomposition of an element of $\mathcal{T}^+$ into elements of $\mathcal{T}_{1/\ell}$ and eq. (3.12), we obtain immediately the inequalities

$$\|\mathbf{b}\| \leq b^0, \qquad \|\mathbf{b}'\| \leq \ell^{-1} b^0, \qquad \|\mathbf{b}''\| \leq \ell^{-1} b^0, \tag{3.14}$$

which can be interpreted as limitations to the velocity, the angular velocity and the acceleration of a moving frame. The relevance of these inequalities for the motion of a particle connected to the frame are discussed in Chapter 8. It easily follows, using the properties of the norm, that an element $B \in \mathcal{T}_{1/\ell}$ can be decomposed as the sum of elements of $\mathcal{T}^+$ only if all of them are proportional to B. This means that $\mathcal{T}_{1/\ell}$ is the set of all the *extremal elements* of the cone $\mathcal{T}^+$.

3.3 The symmetry group $GL(4, \mathbf{R})$

We have seen in Section 3.2 that, under some conditions, the cone $\mathcal{T}^+$ is completely determined by the value of the fundamental length ℓ. Following refs. [7, 11, 12, 14], we now describe the same cone in a different way, which permits an easier discussion of its properties.

We use the Dirac matrices γ_i (see Section 0.3) in a Majorana representation. They are real and satisfy the conditions

$$\gamma_i^T = -C\gamma_i C^{-1}, \qquad \det \gamma_i = 1, \tag{3.15}$$

where γ_i^T is the transposed matrix and C is a real antisymmetric matrix, determined by this equation up to a numeric factor. It is determined up to the sign if we require that its Pfaffian is equal to -1, namely that

$$2^{-3}\epsilon^{ABCD} C_{AB} C_{CD} = -1, \tag{3.16}$$

a choice that is useful for the following developments and implies some restriction to the representation adopted for the γ-matrices. This condition can be rewritten in any of the following equivalent useful forms

$$2^{-3}\epsilon_{ABCD}(C^{-1})^{AB}(C^{-1})^{CD} = -1, \tag{3.17}$$

$$\begin{gathered}\epsilon_{ABCD} = -C_{AB}C_{CD} - C_{AC}C_{DB} - C_{AD}C_{BC}, \\ \epsilon^{ABCD} = -(C^{-1})^{AB}(C^{-1})^{CD} \\ -(C^{-1})^{AC}(C^{-1})^{DB} - (C^{-1})^{AD}(C^{-1})^{BC}\end{gathered} \tag{3.18}$$

and it implies that

$$\det C = 1. \tag{3.19}$$

In a suitable representation one can put $C = \gamma_0$, but one has to remember that these matrices have a different spinor nature as it is clear if we introduce the spinor covariant and contravariant indices and write $\gamma_i^A{}_B$, C_{AB}, $(C^{-1})^{AB}$. In any case, we choose the sign of C in such a way that $C\gamma^0$ is a positive definite matrix.

We define the real symmetric 4×4 matrices

$$\Gamma_i = \ell^{-1}\gamma_i C^{-1}, \qquad \Gamma_{[ik]} = 2^{-1}(\gamma_i\gamma_k - \gamma_k\gamma_i)C^{-1}. \tag{3.20}$$

and we put

$$b = b^\alpha \Gamma_\alpha. \tag{3.21}$$

We also define the real symmetric matrices

$$\breve{\Gamma}^\alpha = (\Gamma_\alpha)^{-1}, \qquad \breve{\Gamma}^i = \ell C\gamma^i, \qquad \breve{\Gamma}^{[ik]} = -2^{-1}C(\gamma^i\gamma^k - \gamma^k\gamma^i), \tag{3.22}$$

and from the property

$$2^{-2}\mathrm{Tr}\,(\breve{\Gamma}^\alpha\Gamma_\beta) = \delta^\alpha_\beta, \tag{3.23}$$

we obtain the inverse formula

$$b^\alpha = 2^{-2}\mathrm{Tr}\,(\breve{\Gamma}^\alpha b). \tag{3.24}$$

We define $\mathcal{T}^+$ by requiring that the real symmetric matrix b is positive semidefinite, namely that

$$\psi^T b \psi \geq 0 \tag{3.25}$$

for any choice of the real spinor ψ. It is clear that $\mathcal{T}^+$ is symmetric under the transformations

$$b \to aba^T, \qquad a \in GL(4, \mathbf{R}). \tag{3.26}$$

Note that a and $-a$ give rise to the same transformation of $\mathcal{T}$. This transformation property means that the matrix b^{AB} represents a contravariant symmetric $GL(4, \mathbf{R})$ spinor. The possible physical relevance of many-dimensional spaces with a local symmetry group different from a many-dimensional Lorentz group has been examined, in a different context, in ref. [85]

Since a complex 2×2 matrix can be considered as a real 4×4 matrix, the 16-dimensional symmetry group $GL(4, \mathbf{R})$ contains a subgroup isomorphic to $SL(2, \mathbf{C})$, namely to the universal covering of the proper orthochronous Lorentz group. The infinitesimal Lorentz transformations are given by

$$a \sim 1 + 2^{-1}\zeta^{[ik]}\Sigma_{[ik]}, \tag{3.27}$$

where the matrices $\Sigma_{[ik]}$ are given by eq. (1.17). From this formula we obtain, as it was expected,

$$\delta b^i = \zeta^i{}_j b^j, \qquad \delta b^{[ik]} = \zeta^i{}_j b^{[jk]} + \zeta^k{}_j b^{[ij]}. \tag{3.28}$$

Since $\mathcal{T}^+$ is a Lorentz invariant closed cone with nonempty interior, it coincides with the cone defined in Section 3.2 and we have only to show that the the parameter ℓ is the same. It is sufficient to remark that the extremal elements defined by eq. (3.12) with $a = \ell^{-1}$ belong to the boundary of $\mathcal{T}^+$ and on this boundary the determinant $\det b$ vanishes.

By means of a direct calculation, using an explicit representation of the gamma matrices, we obtain

$$\begin{aligned}\det b = {} & (\ell^{-2}(b^0)^2 - \ell^{-2}\|\mathbf{b}\|^2 - \|\mathbf{b}'\|^2 - \|\mathbf{b}''\|^2)^2 - 4\ell^{-2}\|\mathbf{b} \times \mathbf{b}'\|^2 \\ & -4\|\mathbf{b}' \times \mathbf{b}''\|^2 - 4\ell^{-2}\|\mathbf{b}'' \times \mathbf{b}\|^2 + 8\ell^{-2} b^0\, \mathbf{b} \cdot \mathbf{b}' \times \mathbf{b}''.\end{aligned} \tag{3.29}$$

A simple substitution shows that the required condition is satisfied, if the parameter ℓ is the same as the one defined in Section 3.2.

It may be stimulating to propose a physical interpretation of $\det b$ [22]. We consider a moving frame $\tau \to s(\tau)$ with the property

$$\frac{ds(\tau)}{d\tau} = b^\alpha A_\alpha \in \mathcal{T}^+ \tag{3.30}$$

and we consider the integral

$$\ell \int_{\tau_0}^{\tau} (\det b)^{1/4}\, d\tau. \tag{3.31}$$

For $\ell \to 0$ this integral takes the form

$$\int_{\tau_0}^{\tau} ((b^0)^2 - \|\mathbf{b}\|^2)^{1/2}\, d\tau, \tag{3.32}$$

which gives the relativistic formula for the time measured by an *ideal clock* moving with the frame $s(\tau)$ and takes into account the influence of the velocity on the clock rate. It is natural to assume that eq. (3.31) also takes into account the influence of the acceleration and the angular velocity of the frame and describes an *ideal accelerated clock*.

It is clear that all the real clocks are influenced by the inertial forces due to acceleration and by the centrifugal forces due to rotation. These forces can even destroy the clock mechanism. Eq. (3.31) could describe a dependence that cannot be made arbitrarily small by building more and more robust mechanisms and cannot be explained in terms of the normal theory.

The function $d(B) = (\det b)^{1/4}$ is homogeneous of degree one and can be considered as a *pseudo-norm* defined on the cone T^+. We choose a representation of the γ matrices in which $C\gamma^0 = 1$ and we indicate by $\beta_1, \ldots, \beta_4$ the positive eigenvalues of b. Then we have

$$d(B) = (\beta_1\beta_2\beta_3\beta_4)^{1/4} \leq 2^{-2}(\beta_1 + \beta_2 + \beta_3 + \beta_4) = 2^{-2}\mathrm{Tr}\,(b) = \ell b^0. \quad (3.33)$$

We can also deduce the interestig equation

$$d(B + B') \geq d(B) + d(B'). \quad (3.34)$$

Note the different inequality sign with respect to the familiar "triangular" property of a norm. Since the determinants are invariant under $SL(4, \mathbf{R})$, we can apply a transformation of this group in such a way that $b + b'$ is a multiple of the unit matrix. It follws that

$$d(B + B') = 2^{-2}\mathrm{Tr}\,(b + b') = 2^{-2}(\mathrm{Tr}\,b + \mathrm{Tr}\,b') \geq d(B) + d(B'). \quad (3.35)$$

There is an analogy with the pseudo-norm of a relativistic 4-vector, defined on the future cone by the formula

$$d(v) = (-v^i v_i)^{1/2}. \quad (3.36)$$

It satisfies an inequality similar to eq. (3.34). In the tangent spaces of the spacetime manifold, it is given by the square root of a quadratic form defined by the metric tensor. If the quadratic form were definite positive, $d(v)$ would be a norm and $\mathcal{M}$ would be a Riemannian space, otherwise it is a pseudo-Riemannian space. In the space $\mathcal{S}$ the pseudo-norm is not the square root of a quadratic form, but the fourth root of a form of degree 4. If $d(B)$ were a norm, positive and defined for every vector B, $\mathcal{S}$ would be a Finslerian space [86]. Since this is not the case, it is a *pseudo-Finslerian space*.

It is interesting to remark the double interplay between symmetry groups and cones [21]. The rotational symmetry together with a choice of a maximal valocity determines the future cone in a tangent space of $\mathcal{M}$. The symmetry group of this cone is the product of the Lorentz group and the dilatation group. The Lorentz symmetry together with a choice of a maximal acceleration (or of a fundamental length) determines the cone $\mathcal{T}^+$. The symmetry group of this cone is $GL(4, \mathbf{R})$.

3.4 Orbits in $\mathcal{T}$ and causal influence

The action of $GL(4, \mathbf{R})$ on the linear space $\mathcal{T}$ splits it into orbits, that, according to Sylvester's law of inertia, are characterized by the numbers p and q of the positive and negative eigenvalues of the matrix b. We indicate these orbits by $\mathcal{T}_{pq}$ with $p + q \leq 4$. We have

$$\mathcal{T}_{mn} = -\mathcal{T}_{nm} \tag{3.37}$$

and the closure of one of these orbits is given by the formula

$$\overline{\mathcal{T}}_{pq} = \bigcup_{\substack{m \leq p \\ n \leq q}} \mathcal{T}_{mn}. \tag{3.38}$$

If $p + q = 4$, the orbit $\mathcal{T}_{pq}$ is open and the other orbits are contained in the surface defined by

$$\det b = 0. \tag{3.39}$$

If we fix the quantities $\mathbf{b}, \mathbf{b}', \mathbf{b}''$, the fourth degree equation (3.39) determines four real values of b^0, that divide the real axis into five open (possibly empty) intervals, corresponding to the five open orbits. When $\ell \to 0$, two roots tend to $\|\mathbf{b}\|$ and the other two roots tend to $-\|\mathbf{b}\|$. As a consequence, the open orbits (or their closures) tend to the following limits

$$\begin{gathered}\overline{\mathcal{T}}_{40} \to \{b^0 \geq \|\mathbf{b}\|\}, \qquad \overline{\mathcal{T}}_{31} \to \{b^0 = \|\mathbf{b}\|\}, \qquad \mathcal{T}_{22} \to \{|b^0| < \|\mathbf{b}\|\}, \\ \overline{\mathcal{T}}_{13} \to \{b^0 = -\|\mathbf{b}\|\}, \qquad \overline{\mathcal{T}}_{04} \to \{b^0 \leq -\|\mathbf{b}\|\}.\end{gathered} \tag{3.40}$$

In this limit, the open orbits $\mathcal{T}_{31}$ and $\mathcal{T}_{13}$ and their closures tend to 9-dimensional surfaces on which the four-vector b is lightlike, while the limit of the other open orbits is determined by the requirement that b is spacelike or timelike (with different signs of b^0). The physical interpretation of these limits is the same as in normal relativistic theories.

For $\ell > 0$, the causal structure of $\mathcal{T}$ is more complicated and its physical interpretation is not evident. A discussion of this problem is given in refs. [12, 14]. The interpretation of

$$\overline{\mathcal{T}}_{40} = \mathcal{T}^+, \qquad \overline{\mathcal{T}}_{04} = -\mathcal{T}^+ \tag{3.41}$$

has already been given in Section 3.1 in terms of feasibility.

The cone $\mathcal{T}^+$ defines a partial ordering in $\mathcal{S}$. If $\mathcal{S}$ is flat, namely it is an affine space and all the vector fields A_α commute, we can define the "difference" operation $s' - s \in \mathcal{T}$. Then the order relation $s \leq s'$ is defined by

$$s' - s \in \mathcal{T}^+. \tag{3.42}$$

The convexity of $\mathcal{T}^+$ assures the validity of the transitive property of this relation. In a nonflat space, we say that $s \leq s'$ if there is a curve $\tau \to s(\tau)$ with $s(0) = s$, $s(1) = s'$ and all its tangent vectors belonging to $\mathcal{T}^+$. The relation defined in this way is reflexive and transitive and it is also antisymmetric if there are no closed curves with all the tangent vectors in $\mathcal{T}^+$.

We try an interpretation of the other orbits only when $\mathcal{S}$ is flat. This is a good approximation if we consider a sufficiently small region of $\mathcal{S}$. In Chapter 11 (see also [12]) we show that the free quantum field operators $\Phi(s)$ and $\Phi(s')$ defined in the flat space $\mathcal{S}$ commute if $s' - s \in \mathcal{T}_{22}$. In analogy with the quantum field theories in Minkowski spacetime, we say that in this case the points s and s' are *causally disjoint*, namely there is no *causal influence* between them.

Of course, we have to clarify the concept of causal influence and this requires the discussion of a kind of procedures that generalize the measurement and the transformation procedures discussed in Section 2.2. We call them *restricted measurement procedures* and they have the aim, starting from a local frame s, of measuring some observables and to build a new local frame s'. A class of equivalent restricted measurement procedure is called a *restricted measurement*. Of course, we must have $s \leq s'$.

The result of the restricted measurement is available at the frame s', in the sense that it can be used to specify some detail of a new procedure performed starting from s'. We say that the restricted measurement starts at s and ends at s'. We have already observed in Section 2.6 that it is not reasonable to assume that the result of a measurement performed in a local frame is available immediately to be used in procedures starting from the same frame. The result of an unrestricted measurement is available only after a macroscopic time, to be registered by some macroscopic instrument.

We say that a local frame s' is causally influenced by a local frame s if a procedure starting at s can influence the result of a restricted measurement ending at s'. One cannot justify the assumption that this relation is transitive, because the information transmitted from s to s' is not necessarily of the kind that can be transmitted from s' to s''.

We assume that s' is causally influenced by s if

$$s' - s \in \mathcal{T}^C = \mathcal{T}^+ \cup \overline{\mathcal{T}}_{31} = \bigcup_{q=0,1} \mathcal{T}_{pq}. \tag{3.43}$$

This assumption is consistent with the interpretation of $\mathcal{T}_{22}$, because this orbit is just the complement of $\mathcal{T}^C \cup -\mathcal{T}^C$. It is also consistent with the limit (3.40) and the normal relativity theory. The set $\mathcal{T}^C$ is not convex and the relation of causal influence is not transitive. However, we have

$$\mathcal{T}^C + \mathcal{T}^+ = \mathcal{T}^C. \tag{3.44}$$

If

$$s' - s \in \mathcal{T}^C \cap -\mathcal{T}^C = \overline{\mathcal{T}}_{11} \tag{3.45}$$

s and s' can influence one another in both directions. We call this relation *reciprocal influence*. In a normal theory this can happen only if the two local frames belong to the same fiber and, in fact, for $\ell \to 0$ we have

$$\overline{\mathcal{T}}_{11} \to \{b = 0\} = \mathcal{T}_V. \tag{3.46}$$

The reciprocal influence replaces the spacetime coincidence of the normal relativistic theories in the same way as the spacelike separation of two events replaces in a relativistic theory the time coincidence (simultaneity) of the Newtonian theory. In a situation in which the light velocity can be considered as infinite, there is no observable difference between spacelike separation and simultaneity. In a similar way, in a situation in which one can disregard the fundamental length, there is no observable difference between reciprocal influence and spacetime coincidence.

Note that the relation of reciprocal influence (as well as the relation of spacelike separation) is not transitive and cannot be used to build equivalence classes, to be interpreted as points of spacetime (events).

From the property (3.37) we see that only the orbits $\mathcal{T}_{11}$ and $\mathcal{T}_{22}$ contain straight lines. The orbit $\mathcal{T}_{22}$ also contains 3-dimensional vector subspaces, for instance the subspace generated by A_1, A_2 and A_3. Though we have no proof, we conjecture that $\mathcal{T}_{22}$ does not contain 4-dimensional vector subspaces.

We say that a submanifold of $\mathcal{S}$ is *spacelike* if all its tangent vectors belong to $\mathcal{T}_{22}$. We shall see in Section 4.1 that important physical quantities (conserved or not) are given by integrals of 3-forms on a 3-dimensional submanifold. It is natural to choose spacelike submanifolds, but a more restrictive choice may be necessary.

3.5 $SL(4,\mathbf{R})$ spinors and $SO(3,3)$ tensors

The symmetry group $\mathcal{F}^G$ introduced in Section 2.5 leaves the geometry of $\mathcal{T}$ and in particular the cone $\mathcal{T}^+$ invariant and therefore it must be a subgroup of the 16-dimensional group $GL(4,\mathbf{R})$. In the present Section we study some tensor representations of this group and of its subgroup $SL(4,\mathbf{R})$. In order to avoid confusion with the tensors of the Lorentz group and of other pseudo-orthogonal groups, we use the word *spinors*.

We have already met the symmetric contravariant spinor b^{AB}, which determines an element of the vector space $\mathcal{T}$. In a similar way one can describe the 10-momentum introduced in Section 1.9 by means of a symmetric covariant spinor p_{AB} that transforms according to a different inequivalent representation of $SL(4,\mathbf{R})$. It is given by

$$p = p_\alpha \breve{\Gamma}^\alpha, \qquad p_\alpha = 2^{-2}\mathrm{Tr}\,(\Gamma_\alpha p) \tag{3.47}$$

and we have

$$b^\alpha p_\alpha = b^i p_i + 2^{-2} b^{[ik]} p_{[ik]} = 2^{-2}\mathrm{Tr}\,(bp). \tag{3.48}$$

The determinant of the matrix p which is invariant under $SL(4,\mathbf{R})$, can be written in the form

$$\begin{aligned}\det p = {} & \left(\ell^2 (p_0)^2 - \ell^2\|\mathbf{p}\|^2 - \|\mathbf{p}'\|^2 - \|\mathbf{p}''\|^2\right)^2 - 4\ell^2\|\mathbf{p}\times\mathbf{p}'\|^2 \\ & -4\|\mathbf{p}'\times\mathbf{p}''\|^2 - 4\ell^2\|\mathbf{p}''\times\mathbf{p}\|^2 - 8\ell^2 p_0\, \mathbf{p}\cdot\mathbf{p}'\times\mathbf{p}''\end{aligned} \tag{3.49}$$

where we have introduced the 3-dimensional vector notation (1.74)

In the following, we shall also meet antisymmetric spinors. A covariant antisymmetric spinor f_{AB} can always be written in the form

$$f = f_u \breve{\Theta}^u, \qquad f_u = -2^{-2}\mathrm{Tr}\,(\Theta_u f), \tag{3.50}$$

where $u = 0, \ldots, 5$. We have introduced the antisymmetric matrices

$$\Theta_i = \gamma_i\gamma_5 C^{-1}, \qquad \Theta_4 = -C^{-1}, \qquad \Theta_5 = -\gamma_5 C^{-1} = -G^{-1}. \tag{3.51}$$

$$\breve{\Theta}^u = -(\Theta_u)^{-1}, \qquad \breve{\Theta}^i = C\gamma^i\gamma^5, \qquad \breve{\Theta}^4 = C, \qquad \breve{\Theta}^5 = C\gamma^5 = G. \tag{3.52}$$

The matrix γ_5 is defined in Section 0.3. In the Majorana representation it is real and has the property

$$\gamma_5^T = C\gamma_5 C^{-1}. \tag{3.53}$$

We have

$$-2^{-2}\mathrm{Tr}\,(\breve{\Theta}^u\Theta_v) = \delta^u_v. \tag{3.54}$$

In a similar way, a contravariant antisymmetric spinor h^{AB} can always be written in the form

$$h = h^u\Theta_u, \qquad h^u = -2^{-2}\mathrm{Tr}\,(\breve{\Theta}^u h). \tag{3.55}$$

The group $GL(4,\mathbf{R})$ has two connected components with different signs of $\det a$. In the following we treat with some detail only the connected component of the identity $GL(4,\mathbf{R})_0$. The 15-dimensional subgroup $SL(4,\mathbf{R}) \subset GL(4,\mathbf{R})_0$ is defined by the condition $\det a = 1$ and is connected. The totally antisymmetric spinors ϵ^{ABCD} and ϵ_{ABCD} are invariant under this subgroup. They define an invariant (not positive) quadratic form (namely the Pfaffian) in the spaces of the covariant and contravariant antisymmetric spinors and $SL(4,\mathbf{R})$ acts on these spaces by means of pseudo-orthogonal transformations.

By means of eq. (3.18), we obtain

$$2^{-3}\epsilon^{ABCD}\breve{\Theta}^u_{AB}\breve{\Theta}^v_{CD} = g^{uv}, \qquad 2^{-3}\epsilon_{ABCD}\Theta^{AB}_u\Theta^{CD}_v = g_{uv}, \tag{3.56}$$

$$2^{-3}\epsilon^{ABCD}f_{AB}f_{CD} = g^{uv}f_uf_v, \qquad 2^{-3}\epsilon_{ABCD}h^{AB}h^{CD} = g_{uv}h^uh^v, \tag{3.57}$$

where $g^{uv} = g_{uv}$ is the diagonal 6-dimensional metric tensor with $g^{11} = g^{22} = g^{33} = 1$, $g^{00} = g^{44} = g^{55} = -1$. The representation of $SL(4,\mathbf{R})$ in the space of the antisymmetric spinors defines a homomorphism $SL(4,\mathbf{R}) \to O(3,3)$. Since both these Lie groups have dimension 15, this is a local isomorphism and the Lie algebras $sl(4,\mathbf{R})$ and $o(3,3)$ are isomorphic. The group $SL(4,\mathbf{R})$ is mapped onto the connected component of the identity $SO(3,3)_0 \subset SO(3,3)$ and the kernel of the homomorphism is composed of the two matrices ± 1.

We can write the infinitesimal transformations of $SL(4,\mathbf{R})$ in the form

$$a \sim 1 + 2^{-1}\zeta^{[uv]}\Sigma_{[uv]}, \qquad u, v = 0,\ldots,5, \tag{3.58}$$

where the 4×4 real matrices $\Sigma_{[ik]}$ are given by eq. (1.17) and

$$\Sigma_{[i4]} = -\Sigma_{[4i]} = 2^{-1}\gamma_i\gamma_5, \qquad \Sigma_{[i5]} = -\Sigma_{[5i]} = 2^{-1}\gamma_i,$$
$$\Sigma_{[45]} = -\Sigma_{[54]} = 2^{-1}\gamma_5. \tag{3.59}$$

They provide a basis for the Lie algebra $sl(4, \mathbf{R}) = so(3,3)$ and satisfy the commutation relations

$$[\Sigma_{[uv]}, \Sigma_{[xy]}] = g_{vx}\Sigma_{[uy]} - g_{ux}\Sigma_{[vy]} - g_{vy}\Sigma_{[ux]} + g_{uy}\Sigma_{[vx]}. \tag{3.60}$$

Note that the elements of the Lie algebra, on which the adjoint representation of the group operates, can be represented by traceless mixed rank two spinors or by antisymmetric rank two tensors.

The infinitesimal transformations of Dirac spinors and of covariant and contravariant antisymmetric spinors are given by

$$\delta\Psi = 2^{-1}\zeta^{[uv]}\Sigma_{[uv]}\Psi, \tag{3.61}$$

$$\delta f = -2^{-1}\zeta^{[uv]}(\Sigma^T_{[uv]}f + f\Sigma_{[uv]}), \qquad \delta h = 2^{-1}\zeta^{[uv]}(\Sigma_{[uv]}h + h\Sigma^T_{[uv]}). \tag{3.62}$$

One can easily check the formulas

$$\begin{aligned} \Sigma_{[uv]}\Theta_w + \Theta_w\Sigma^T_{[uv]} &= g_{vw}\Theta_u - g_{uw}\Theta_v, \\ \Sigma^T_{[uv]}\breve{\Theta}^w + \breve{\Theta}^w\Sigma_{[uv]} &= \delta^w_u\breve{\Theta}_v - \delta^w_v\breve{\Theta}_u \end{aligned} \tag{3.63}$$

and from eqs. (3.50) and (3.55) one obtains the expected 6-vector infinitesimal transformations

$$\delta f_u = \zeta_u{}^v f_v, \qquad \delta h^u = \zeta^u{}_v h^v. \tag{3.64}$$

The infinitesimal transformations of covariant and contravariant symmetric spinors are

$$\delta p = -2^{-1}\zeta^{[uv]}(\Sigma^T_{[uv]}p + p\Sigma_{[uv]}), \qquad \delta b = 2^{-1}\zeta^{[uv]}(\Sigma_{[uv]}b + b\Sigma^T_{[uv]}). \tag{3.65}$$

In order to introduce the corresponding $SO(3,3)$ tensors, we define the quantities

$$p_{[uvw]} = 2^{-2}\mathrm{Tr}\,(\Theta_u\ \breve{\Theta}_v\Theta_w p), \qquad b^{[uvw]} = 2^{-2}\mathrm{Tr}\,(\breve{\Theta}^u\ \Theta^v\breve{\Theta}^w b). \tag{3.66}$$

It is easy to show that they transform as tensors of rank 3, namely

$$\begin{aligned} \delta p_{[uvw]} &= \zeta_u{}^x f_{[xvw]} + \zeta_v{}^x f_{[uxw]} + \zeta_w{}^x f_{[uvx]}, \\ \delta b^{[uvw]} &= \zeta^u{}_x b^{[xvw]} + \zeta^v{}_x b^{[uxw]} + \zeta^w{}_x b^{[uvx]}. \end{aligned} \tag{3.67}$$

By computing the traces, we find that, as it is suggested by the notation, these quantities are antisymmetric with respect to their three indices and are given by

$$\begin{aligned} p_{[i45]} &= -\ell p_i \qquad & p_{[ijk]} &= -\ell\epsilon_{ijk}{}^l p_l, \\ p_{[ik4]} &= p_{[ik]}, \qquad & p_{[ik5]} &= 2^{-1}\epsilon_{ik}{}^{jl}p_{[jl]}, \end{aligned} \tag{3.68}$$

$$
\begin{aligned}
b^{[i45]} &= -\ell^{-1} b^i \qquad & b^{[ijk]} &= \ell^{-1}\epsilon^{ijk}{}_l b^l, \\
b^{[ik4]} &= b^{[ik]}, \qquad & b^{[ik5]} &= -2^{-1}\epsilon^{ik}{}_{jl} b^{[jl]}.
\end{aligned} \tag{3.69}
$$

We enclose an antisymmetric set of three indices into square brackets to indicate that they replace a greek index that labels a basis in the space $\mathcal{T}^*$ or $\mathcal{T}$. In a similar way we define the vector fields $A_{[uvw]}$ and the 1-forms $\omega^{[uvw]}$.

The general 6-dimensional antisymmetric tensors of rank 3 have 20 independent components and, in order to describe the vectors of a 10-dimensional space, they must satisfy some constraint. In fact, from the formulas given above, we have

$$
6^{-1}\epsilon_{uvw}{}^{xyz} p_{[xyz]} = p_{[uvw]}, \qquad 6^{-1}\epsilon^{uvw}{}_{xyz} b^{[xyz]} = -b^{[uvw]}. \tag{3.70}
$$

This means that these tensors are respectively self-dual and anti-self-dual.

In terms of the usual Lorentz components, the transformation formulas (3.67) take the form

$$
\begin{aligned}
\delta p_i &= \zeta_i{}^j p_j + \ell^{-1}\zeta^{[k5]} p_{[ik]} - 2^{-1}\ell^{-1}\zeta^{[j4]}\epsilon_{ij}{}^{kl} p_{[kl]}, \\
\delta p_{[ik]} &= \zeta_i{}^j p_{[jk]} + \zeta_k{}^j p_{[ij]} + \ell(\zeta_{[i5]} p_k - \zeta_{[k5]} p_i) \\
&\quad -\ell\zeta^{[j4]}\epsilon_{ikj}{}^l p_l - 2^{-1}\zeta^{[45]}\epsilon_{ik}{}^{jl} p_{[jl]}
\end{aligned} \tag{3.71}
$$

$$
\begin{aligned}
\delta b^i &= \zeta^i{}_j b^j + \ell\zeta_{[k5]} b^{[ik]} - 2^{-1}\ell\zeta^{[j4]}\epsilon^i{}_{jkl} b^{[kl]}, \\
\delta b^{[ik]} &= \zeta^i{}_j b^{[jk]} + \zeta^k{}_j b^{[ij]} + \ell^{-1}(\zeta^{[i5]} b^k - \zeta^{[k5]} b^i) \\
&\quad -\ell^{-1}\zeta^{[j4]}\epsilon^{ik}{}_{jl} b^l + 2^{-1}\zeta^{[45]}\epsilon^{ik}{}_{jl} b^{[jl]}.
\end{aligned} \tag{3.72}
$$

These formulas can be interpreted as the transformations of the components corresponding to the infinitesimal transformations of the basis vectors A_α and ω^α in the vector spaces $\mathcal{T}$ and $\mathcal{T}^*$. It follows that A_α and ω^α transform, respectively, in the same way as the components p_α and b^α.

3.6 Subgroups of $SL(4, \mathbf{R})$

A subgroup of $GL(4, \mathbf{R})$ that leaves a nondegenerate antisymmetric covariant spinor f invariant, namely

$$
a^T f a = f, \qquad \det f \neq 0, \tag{3.73}
$$

is contained in $SL(4,\mathbf{R})$. It is isomorphic to the symplectic group $Sp(4,\mathbf{R})$ and locally isomorphic to the anti-de Sitter group $SO(2,3)$, namely to the subgroup of $SO(3,3)$ that leaves the 6-vector f_u invariant.

All these symplectic subgroups are reciprocally conjugate and we shall use only the ones that contain the subgroup $SL(2,\mathbf{C})$ defined by eq. (3.27). They are defined by the condition (3.73) with

$$f = \cos\alpha C + \sin\alpha G, \qquad -\pi/2 < \alpha \leq \pi/2 \tag{3.74}$$

and we indicate them by $Sp(4,\mathbf{R})_\alpha$. In particular, we call $Sp(4,\mathbf{R})_0 = Sp(4,\mathbf{R})_V$ the *vector symplectic subgroup*, because its Lie algebra contains, besides the six independent generators $\Sigma_{[ik]}$ of $SL(2,\mathbf{C})$, the four elements $\Sigma_{[i5]}$ that transform as the components of a 4-vector. In a similar way we call $Sp(4,\mathbf{R})_{\pi/2} = Sp(4,\mathbf{R})_A$ the *axial symplectic subgroup*, because its Lie algebra contains, besides the generators of $SL(2,\mathbf{C})$, the four elements $\Sigma_{[i4]}$ that transform as the components of an axial vector. We indicate by $SO(2,3)_V$ and $SO(2,3)_A$ the corresponding anti-de Sitter groups or, more exactly, their identity connected components. They leave invariant, respectively, the components f_4 and f_5 of a 6-dimensional vector f_u.

We indicate by $\mathcal{F}_0$ the subgroup of $GL(4,\mathbf{R})$ containing the transformations that do not mix the vertical and the horizontal subspaces and do not involve the parameter ℓ. It is defined by the condition

$$a\gamma_5 = \pm\gamma_5 a \tag{3.75}$$

and it contains the subgroup $SL(2,\mathbf{C})$, the space reflection represented by $a = \pm\gamma_0$, the dilatations of $\mathcal{T}$ and the one-parameter subgroup $U(1)_5$ generated by $\Sigma_{[45]}$ and described by the formulas

$$a = \exp(2^{-1}\alpha\gamma_5), \tag{3.76}$$

$$b^i \to b^i, \qquad b^{[ik]} \to \cos(\alpha) b^{[ik]} + 2^{-1}\sin(\alpha)\epsilon^{ik}{}_{jl} b^{[jl]}. \tag{3.77}$$

In particular, for $\alpha = \pm\pi$ we obtain the *vertical reflection*, represented by $a = \pm\gamma_5$, which changes the sign of the vertical vectors leaving the horizontal vectors unchanged. Note that

$$\begin{aligned} \gamma_0^T C\gamma_0 = C, \qquad \gamma_0^T G\gamma_0 = -G, \\ \gamma_5^T C\gamma_5 = -C, \qquad \gamma_5^T G\gamma_5 = -G, \end{aligned} \tag{3.78}$$

and therefore

$$\gamma_0 \in Sp(4, \mathbf{R})_V, \qquad \gamma_0\gamma_5 \in Sp(4, \mathbf{R})_A. \tag{3.79}$$

Since these groups are connected, a symmetry with respect to γ_0 or $\gamma_0\gamma_5$ follows from the symmetry with respect to infinitesimal transformations of the corresponding group. Since space inversion is not a symmetry of nature, this remark gives an argument in favour of of the choice of $Sp(4, \mathbf{R})_A$ as a high symmetry group, as it was observed in refs. [6] and [12].

If, after the rescaling

$$\ell^{-1}\zeta^{[i4]} = \zeta^i_A, \qquad \ell^{-1}\zeta^{[i5]} = \zeta^i_V \tag{3.80}$$

of the parameters, we perform the limit $\ell \to 0$, we obtain a *contraction* [87, 88] of the group $GL(4, \mathbf{R})$. The subgroup $\mathcal{F}_0$, that does not involve ℓ is not affected by the the contraction. The contracted group is a semi-direct product of $\mathcal{F}_0$ and a 8-dimensional commutative group parametrized by ζ^i_A and ζ^i_V.

By taking the limit of eqs. (3.71) and (3.72), we find the transformation laws with respect to the commutative subgroup of the contracted group

$$p_i \to p_i + \zeta^k_V p_{[ik]} - 2^{-1}\zeta^j_A \epsilon_{ij}{}^{kl} p_{[kl]}, \qquad p_{[ik]} \to p_{[ik]}, \tag{3.81}$$

$$b^i = b^i, \qquad b^{[ik]} \to b^{[ik]} + \zeta^i_V b^k - \zeta^k_V b^i - \zeta^j_A \epsilon^{ik}{}_{jl} b^l, \tag{3.82}$$

that also describe finite transformations. Note that the contracted group is a symmetry group of the normal wedge defined by eq. (3.5). The complete symmetry group of this wedge, however, is much larger.

In order to introduce the fundamental length ℓ by means of a symmetry group, in the same way as one introduces the fundamental velocity assuming the Lorentz symmetry, one has to use at least one of the symplectic groups $Sp(4, \mathbf{R})_\alpha$. We shall see in Chapter 7 that $Sp(4, \mathbf{R})_A$ is the best choice. This group is locally isomorphic to the anti-de Sitter group $SO(2, 3)_A$, and it is possible to develop a 5-dimensional tensor formalism based on this group, which was introduced in ref. [6].

We have already seen in Section 3.5 that an antisymmetric covariant spinor $f_{[AB]}$ is equivalent to a 6-dimensional $SO(3, 3)$ vector f_u. If we consider only the subgroup $SO(2, 3)_A$, the quantities f_u, $u = 0, \ldots, 4$ are the components of a 5-dimensional vector and f_5 is a scalar. We have also seen that a contravariant symmetric spinor $b^{(AB)}$ is equivalent to a 6-dimensional self-dual antisymmetric tensor $b^{[uvw]}$, that is completely determined by its components of the form $b^{[uv5]}$, with $u, v = 0, \ldots, 4$. These components transform as a 5-dimensional antisymmetric tensor of rank 2 under $SO(2,3)_A$, since the last index is not affected by this group. In a similar way, a contravariant spinor $h_{[AB]}$ is equivalent to a 5-dimensional antisymmetric tensor of rank 2 $h_{[uv5]}$.

In the transition from the 6-dimensional to the 5-dimensional tensor formalism, the following consequences of eq. (3.70) are useful:

$$p_{[uvw]} = 2^{-1}\epsilon_{uvw}{}^{xy} p_{[xy5]}, \qquad b^{[uvw]} = 2^{-1}\epsilon^{uvw}{}_{xy} b^{[xy5]}, \\ u, v, w, x, y = 0, \ldots, 4. \tag{3.83}$$

3.7 The spinor representation of the structure coefficients

All the geometric quantities introduced in Section 1.7 can be written in spinor form by means of a change of basis in the spaces $\mathcal{T}$ and $\mathcal{T}^*$. For instance we put

$$A_{(AB)} = \breve{\Gamma}^{\alpha}_{(AB)} A_{\alpha}, \qquad \omega^{(AB)} = \Gamma^{(AB)}_{\alpha} \omega^{\alpha}, \tag{3.84}$$

$$F^{(EF)}_{(AB)(CD)} = \breve{\Gamma}^{\alpha}_{(AB)} \Gamma^{(EF)}_{\gamma} \breve{\Gamma}^{\beta}_{(CD)} F^{\gamma}_{\alpha\beta}. \tag{3.85}$$

We enclose a symmetric pair of indices into round brackets to indicate that they replace a greek index.

The structure coefficients for a spacetime with constant curvature, given by eqs. (1.12), (1.36) and (1.78) take the form

$$\begin{aligned} \hat{F}^{(EF)}_{(AB)(CD)} &= 4(3 - \ell^2\rho)\, \delta^{(E}_{(A}\, C_{B)(C}\, \delta^{F)}_{D)} \\ &-4(1 + \ell^2\rho)\, (\gamma_5)^{(E}{}_{(A}\, G_{B)(C}\, \delta^{F)}_{D)} \\ &-4(1 + \ell^2\rho)\, \delta^{(E}_{(A}\, G_{B)(C}\, (\gamma_5)^{F)}{}_{D)} \\ &+4(1 + \ell^2\rho)\, (\gamma_5)^{(E}{}_{(A}\, C_{B)(C}\, (\gamma_5)^{F)}{}_{D)}. \end{aligned} \tag{3.86}$$

where the round brackets indicate the symmetrization of the enclosed indices.

This expression contains the spinors C and γ_5 and it is not invariant with respect to the whole group $GL(4, \mathbf{R})$ and not even with respect to its symplectic subgroups. It is invariant with respect to a subgroup isomorphic to $SL(2, \mathbf{C})$, namely it is Lorentz invariant, as it is evident from the original definition given by eqs. (1.12), (1.36) and (1.78). Actually, γ_5 appears an even number of times and the structure constants are also invariant under space reflection. Their sign changes under vertical reflection.

For $\rho = -\ell^{-2}$, γ_5 disappears from the expression (3.86) and the structure constants are invariant with respect to the vector symplectic subgroup $Sp(4,R)_V$ locally isomorphic to an anti-de Sitter subgroup. These structure constants describe a space-time with a very large constant negative curvature. It has no physical relevance, not even as a highly unstable inflationary vacuum, due to the wrong sign of the curvature $R = 12\rho$.

The contractions of the expressions (3.86) with respect to one or two pairs of indices are given by the useful formulas

$$\hat{F}^{(EF)}_{(AB)(CF)} = 6(3-\ell^2\rho)\,\delta^E_{(A}C_{B)C} - 6(1+\ell^2\rho)\,(\gamma_5)^E{}_{(A}G_{B)C}, \tag{3.87}$$

$$\hat{F}^{(EF)}_{(AE)(FC)} = 12(4-\ell^2\rho)\,C_{AC}. \tag{3.88}$$

An interestig linear function of the structure coefficients is the antisymmetric spinor

$$48t_{AB} = F^{(CD)}_{(AC)(DB)}, \tag{3.89}$$

which can be written in matrix form as

$$\begin{aligned} 48t = {}& \ell C\gamma^i\gamma_m\gamma^j F^m_{ij} + 2^{-1}\ell^2 C\gamma^i\gamma_m\gamma_n\gamma^j F^{[mn]}_{ij} \\ & -2^{-1}C(\gamma^i\gamma^k\gamma_m\gamma^j - \gamma^j\gamma_m\gamma^i\gamma^k)F^m_{[ik]j} \\ & -2^{-2}\ell C(\gamma^i\gamma^k\gamma_m\gamma_n\gamma^j - \gamma^j\gamma_m\gamma_n\gamma^i\gamma^k)F^{[mn]}_{[ik]j} \\ & +2^{-2}\ell^{-1}C\gamma^i\gamma^k\gamma_m\gamma^j\gamma^l F^m_{[ik][jl]} + 2^{-3}C\gamma^i\gamma^k\gamma_m\gamma_n\gamma^j\gamma^l F^{[mn]}_{[ik][jl]}. \end{aligned} \tag{3.90}$$

From eq. (3.50), by computing the traces, we can obtain the components of the corresponding 6-dimensional vector t_u, which are rather long expressions. In the case of a spacetime with constant curvature we have

$$t_i = 0, \qquad t_4 = 1 - (48)^{-1}\ell^2 R, \qquad t_5 = 0. \tag{3.91}$$

In the general case, we prefer to use a different method based on the 6 and 5-dimensional tensor calculus. The component t_5 is invariant under $SO(2,3)_A$. The structure coefficients are the components of a 5-dimensional tensor of the kind $F^{[yz5]}_{[uv5][wx5]}$ and, by contraction of the indices, one can obtain only one linear $SO(2,3)_A$ invariant, namely we have

$$48t_5 = g^{vw}F^{[xy5]}_{[xv5][wy5]}. \tag{3.92}$$

The choice of the numerical coefficient will soon be justified.

In a similar way, one can define the following 5-dimensional vectors depending linearly on the structure coefficients:

$$\begin{aligned}48 f_u &= -\epsilon_{yz}{}^{vwx} F^{[yz5]}_{[uv5][wx5]},\\ 48 f'_u &= \epsilon_{uy}{}^{vwx} F^{[yz5]}_{[vw5][xz5]},\\ 48 f''_u &= \epsilon_{uxy}{}^{vw} g^{zz'} F^{[xy5]}_{[vz5][z'w5]},\end{aligned} \tag{3.93}$$

where ϵ^{uvwxy} is the 5-dimensional completely antisymmetric tensor. We have

$$\begin{aligned}&\epsilon_{yzvwx} F^{[yz5]}_{[uv'5][w'x'5]} g^{vv'} g^{ww'} g^{xx'} - 2\epsilon_{yuvwx} F^{[yz5]}_{[zv'5][w'x'5]} g^{vv'} g^{ww'} g^{xx'}\\ &-2\epsilon_{yzvwu} F^{[yz5]}_{[xv'5][w'x'5]} g^{vv'} g^{ww'} g^{xx'} = 0,\end{aligned} \tag{3.94}$$

because this expression has been antisymmetrized with respect to the 6 indices u, y, z, v, w, x, which can take only 5 values. In this way we obtain

$$-f_u + 2f'_u - 2f''_u = 0 \tag{3.95}$$

and we see that there are only two independent 5-dimensional vectors that are linear functions of the structure coefficients.

The other components t_u with $u = 0, \ldots, 4$ can be determined starting from t_5 by means of the transformation property

$$\begin{aligned}48\zeta_5{}^u t_u = 48\delta t_5 &= g^{vw} \zeta_5{}^u F^{[xy5]}_{[xvu][wy5]}\\ &+ g^{vw} \zeta_5{}^u F^{[xy5]}_{[xv5][wyu]} + g^{vw} \zeta^5{}_u F^{[xyu]}_{[xv5][wy5]}.\end{aligned} \tag{3.96}$$

Since the infinitesimal parameters $\zeta_5{}^u$ are arbitrary, we obtain, also using eq. (3.83),

$$t_u = f'_u - 2^{-1} f''_u = 2^{-2} f_u + 2^{-1} f'_u. \tag{3.97}$$

The quantities defined above can be expressed as 4-dimensional tensors by means of eqs. (3.68) and (3.69) and we obtain

$$\begin{aligned}48 t_i &= \ell \epsilon_{il}{}^{jk} F^l_{jk} + 2\ell \epsilon_{im}{}^{jl} F^{[km]}_{[kj]l} + 2^{-1} \ell \epsilon^{km}{}_{jl} F^{[jl]}_{[ik]m} - 2^{-2} \ell \epsilon^{jk}{}_{lm} F^{[lm]}_{[jk]i}\\ &-2^{-1} \ell \epsilon_{im}{}^{jl} F^{[km]}_{[jl]k} - 2^{-1} \ell^{-1} \epsilon^{kjl}{}_m F^m_{[ik][jl]} + 2^{-1} \ell^{-1} \epsilon_i{}^{jlm} F^k_{[jl][km]},\\ 48 t_4 &= \ell^2 F^{[ik]}_{ik} + 2g^{kj} F^i_{[ik]j} + g^{kj} F^{[il]}_{[ik][jl]},\\ 48 t_5 &= -2^{-1} \ell^2 \epsilon^{ik}{}_{jl} F^{[jl]}_{ik} - \epsilon^{ikj}{}_l F^l_{[ik]j}\\ &-2^{-1} \epsilon_{mn}{}^{il} g^{kj} F^{[mn]}_{[ik][jl]} - \epsilon_n{}^{kjl} F^{[in]}_{[ik][jl]}.\end{aligned} \tag{3.98}$$

$$48f_i = -\ell\epsilon^{jk}{}_{lm}F^{[lm]}_{[jk]i} - 2\ell\epsilon_{im}{}^{jl}F^{[km]}_{[jl]k} + 2\ell^{-1}\epsilon_i{}^{jlm}F^k_{[jl][km]},$$
$$48f_4 = 4\ell^2 F^{[ik]}_{ik} + 4g^{kj}F^i_{[ik]j}, \tag{3.99}$$

In particular, if we consider a bundle of frames, for instance a solution of the Einstein-Cartan equations, we have

$$t_i = (48)^{-1}\ell\epsilon_{il}{}^{jk}F^l_{jk}, \qquad t_4 = 1 - (48)^{-1}\ell^2 R, \qquad t_5 = 0, \tag{3.100}$$

$$f_i = 0, \qquad f_4 = 1 - (12)^{-1}\ell^2 R. \tag{3.101}$$

In conclusion, the 5-vectors obtained linearly from the structure coefficients are linear cobinations of t_u and f_u. The first 5-vector, together with t_5 forms a 6-vector and the second one, which was introduced in ref. [6], is characterized by the property that it does not depend on the torsion.

We have seen that, if the geometric symmetry group $\mathcal{F}^G$ of the field equations is larger than the orthochronous Lorentz group, the vacuum solution has a lower symmetry, namely we have a *spontaneous symmetry breaking*. In Chapter 7 we shall consider field equations symmetric with respect to the axial symplectic group $Sp(4, \mathbf{R})_A$ and the symmetry breaking can be attributed to a nonvanishing asymptotic value of the 5-vector f_u, which plays a role similar to the role played by the Higgs field in the Standard Model of elementary particles [41].

3.8 Subgroups of $GL(4, \mathbf{R})$ as gauge groups?

In the present notes we always consider the symmetry group $\mathcal{F}^G$ acting on the tangent spaces of $\mathcal{S}$ as a global symmetry, namely the transformation acting on $T_s\mathcal{S}$ does not depend on s and can be considered as a transformation of the linear space $\mathcal{T}$. Of course, it must preserve the cone $\mathcal{T}^+$, namely it must be a subgroup of $SL(4, \mathbf{R})$. Note that, while $SL(2, \mathbf{C})$ is considered as a gauge group on the spacetime $\mathcal{M}$ it represents a global symmetry when considered as a subgroup of $\mathcal{F}^G$.

One may ask if $\mathcal{F}^G$ can be considered as a gauge group, namely if a different group element can act on different tangent spaces. Of course one has to give up the absolute parallelism of $\mathcal{S}$ and therefore the operational interpretation of the vector fields described in Section 2.2. This interpetation can be criticized as a realistic description of the physical operations, but it provides a very useful guide for the theoretical intuition. We think that we cannot renounce to this guide in the present stage of the theoretical research.

We note that $\mathcal{S}$ is a generalization of a principal bundle, namely of the mathematical structure that describes a gauge theory defined on the spacetime manifold $\mathcal{M}$. A gauge theory defined on $\mathcal{S}$ would deserve the name of "second gauge theory" in analogy with the denomination "second quantization" used in the old times to indicate the construction of a quantum field theory. Then, why not to introduce a "third gauge theory" and so on?

Chapter 4

Lagrangian dynamics of classical fields

4.1 Conserved forms

A scalar conserved quantity, for instance the electric charge, is described in the spacetime formalism by a four-vector field $J^\mu(x)$, called a *current*. The charge contained in a space region Σ belonging to a spacelike surface $x^0 = t$ is given by

$$q = \int_\Sigma J^0(t, \mathbf{x})(-\det g)^{1/2}\, d^3\mathbf{x} = \int_\Sigma \hat{\tau}, \tag{4.1}$$

where we have introduced the differential 3-form

$$\hat{\tau} = 6^{-1} J^\lambda(x)(-\det g)^{1/2} \epsilon_{\lambda\mu\nu\sigma}\, dx^\mu \wedge dx^\nu \wedge dx^\sigma. \tag{4.2}$$

The conservation law

$$\frac{\partial}{\partial x^\lambda}\left((-\det g)^{1/2} J^\lambda\right) = 0 \tag{4.3}$$

can be written in the simple form $d\hat{\tau} = 0$.

We consider the pull-back of $\hat{\tau}$ on $\mathcal{S}$, given by

$$\tau_\bullet = \pi_* \hat{\tau} = J^i(s)\, \eta_i = T^i_\bullet\, \eta_i, \tag{4.4}$$

where the forms η_i are defined in Section 0.3. The quantities J^i, that we also indicate by $T^i_\bullet$, are the anholonomic components of the current and we assume that the determinant of the matrix e^i_λ is positive. We have used the formula

$$\pi_* dx^\mu = e^\mu_k \omega^k. \tag{4.5}$$

The conservation law takes the form

$$d\tau_\bullet = 0 \tag{4.6}$$

and the charge is given by an expression of the kind

$$q = \int_\Sigma \tau_\bullet, \tag{4.7}$$

where Σ is a three-dimensional submanifold of $\mathcal{S}$ which has a spacelike projection on $\mathcal{M}$. Note that, in order to obtain the correct sign, one has to choose the orientation of the submanifold Σ in such a way that the 3-form $\omega^1 \wedge \omega^2 \wedge \omega^3$ defines a positive density.

If $\tau_\bullet$ has the form (4.4), we say that it describes a *spatially localized quantity*, but $\tau_\bullet$ may have a more general form, depending on all the one-forms ω^α. In this way, as we shall see in Section 5.1, one can describe the energy-momentum the gravitational field, that is known to be "nonlocalized" in the sense that its spatial density depends on the choice of the coordinates or of a tetrad field [28, 31, 32]. The description of conserved quantities by means of 3-forms is valid also when $\mathcal{S}$ is not a fibre bundle and a spacetime manifold cannot be defined. In this case, the choice of the submanifold Σ and of its orientation becomes a delicate problem that we discuss in Capter 7.

In Maxwell's theory the current is given by the formula

$$(-\det g)^{1/2} J^\mu = \frac{\partial}{\partial x^\nu}\left((-\det g)^{1/2} F^{\nu\mu}\right). \tag{4.8}$$

There is no agreement between various textbooks on the sign in this formula, namely on the definition of the electromagnetic field tensor $F_{\mu\nu}$. We indicate by F_{ik} the anholonomic components of the electromagnetic field and we identify them with the structure coefficients $F^\bullet_{ik}$. With our conventions, the connection with the electric and magnetic 3-dimensional vectors is

$$\mathbf{E} = (F_{01}, F_{02}, F_{03}), \qquad \mathbf{B} = (F_{32}, F_{13}, F_{21}). \tag{4.9}$$

It follows that

$$\hat{\tau} = 6^{-1} \frac{\partial}{\partial x^\tau}\left((-\det g)^{1/2} F^{\tau\lambda}\right) \epsilon_{\lambda\mu\nu\sigma}\, dx^\mu \wedge dx^\nu \wedge dx^\sigma = -d\hat{\sigma}, \tag{4.10}$$

where

$$\hat{\sigma} = 2^{-2}(-\det g)^{1/2} F^{\lambda\tau} \epsilon_{\lambda\tau\mu\nu} dx^\mu \wedge dx^\nu. \tag{4.11}$$

The 2-form $-\hat{\sigma}$ is called sometimes the *Maxwell 2-form* [28] and it is related to the dual electromagnetic tensor.

In this case too, we consider the pull-back of $\hat{\sigma}$ on $\mathcal{S}$, given by

$$\sigma_\bullet = \pi_* \hat{\sigma} = 2^{-2} F^\bullet_{ik}(s) \epsilon^{ik}{}_{jl}\, \omega^j \wedge \omega^l \tag{4.12}$$

and we have

$$\tau_\bullet = -d\sigma_\bullet, \qquad q = \int_\Sigma \tau_\bullet = -\int_{\partial\Sigma} \sigma_\bullet. \tag{4.13}$$

This is a generalized form of the Gauss law, that gives the charge as a surface integral of the electric field. We shall see in the following that a similar *generalized Gauss law* holds for other conserved quantities, related to gauge symmetry properties.

4.2 The action principle and the field equations

Now we want to derive the field equations and the conservation laws of a classical field theory from an *action principle.* It will be clear in the treatment of the Noether theorem given in Section 4.3 that, if the conserved quantities are given by integrals of differential 3-forms on 3-dimensional surfaces, the action must be given by an integral of a differential 4-form on a 4-dimensional surface. This important idea has been proposed independently in refs. [3, 4] and [51, 52].

Note the difference with respect to the Kaluza-Klein theory [47, 48] and its generalizations in a d-dimensional space, in which the action is given by a d-dimensional integral. We shall follow, with some simplifications, the treastment of refs. [3, 5].

It is convenient to use as dynamical variables the 1-forms ω^α and the scalar fields Ψ^U, where A is not necessarily a spinor index. Some of the fields Ψ^U may be anticommuting quantities, that, after quantization, become Fermionic fields. Then, the order of some factors is relevant and we define the derivative of a function λ with respect to the quantity Ψ in such a way that

$$\delta\lambda = \delta\Psi \frac{\partial\lambda}{\partial\Psi}. \tag{4.14}$$

In other words, the derivative with respect to an anticommuting variable is an antiderivation of the graded algebra of functions (see Chapter 10).

If we exclude the presence of higher derivatives, the action principle takes the general form

$$\delta \int_S \lambda = 0, \tag{4.15}$$

where the *Lagrangian form* λ (sometimes simply called *Lagrangian*) is given by

$$\lambda = (24)^{-1} \lambda_{\alpha\beta\gamma\delta}(F^{\eta}_{\epsilon\zeta}, A_\epsilon \Psi^U, \Psi^U)\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta. \tag{4.16}$$

The relativity principle, in the form discussed in Section 2.2, requires that λ has no explicit dependence on s and we shall exploit this property in Section 4.3. The condition (4.15) must hold for an arbitrary choice of the compact 4-dimensional integration surface S, provided that the variations $\delta\omega^\eta$ and $\delta\Psi^U$ vanish on the boundary ∂S of S.

We assume first that the variations $\delta\Psi^U$ and $\delta\omega^\eta$ vanish on the surface S, namely we put

$$\delta\Psi^U = \zeta^U(s) f(s), \qquad \delta\omega^\eta = \zeta^\eta_\epsilon(s) f(s) \omega^\epsilon, \tag{4.17}$$

where $\zeta^U(s)$ and $\zeta^\eta_\zeta(s)$ are arbitrary infinitesimal function and the commuting function $f(s)$ vanishes on the surface S. The quantities ζ^U have the same commutation properties as the fields Ψ^U. On the surface S we have, disregarding higher order infinitesimals,

$$\delta\Psi^U = 0, \qquad \delta A_\epsilon \Psi^U = \zeta^U A_\epsilon f, \tag{4.18}$$

$$\delta\omega^\eta = 0, \qquad \delta d\omega^\eta = \zeta^\eta_\epsilon\, df \wedge \omega^\epsilon. \tag{4.19}$$

The last equality can also be written in the form

$$-2^{-1} \delta F^\eta_{\zeta\epsilon}\, \omega^\zeta \wedge \omega^\epsilon = \zeta^\eta_\epsilon A_\zeta f \omega^\zeta \wedge \omega^\epsilon \tag{4.20}$$

namely

$$\delta F^\eta_{\epsilon\zeta} = \zeta^\eta_\epsilon A_\zeta f - \zeta^\eta_\zeta A_\epsilon f. \tag{4.21}$$

The action principle takes the form

$$\int_S \left(\zeta^U A_\epsilon f \frac{\partial\lambda}{\partial A_\epsilon \Psi^U} - 2\zeta^\eta_\zeta A_\epsilon f \frac{\partial\lambda}{\partial F^\eta_{\epsilon\zeta}} \right) = 0, \tag{4.22}$$

and, since $\zeta^U(s)$ and $\zeta^\eta_\zeta(s)$ are arbitrary, we get

$$\left.\frac{\partial\lambda}{\partial A_\epsilon \Psi^U}\right|_S A_\epsilon f = 0, \qquad \left.\frac{\partial\lambda}{\partial F^\eta_{\epsilon\zeta}}\right|_S A_\epsilon f = 0, \tag{4.23}$$

where $|_S$ means the restriction of the differential form to the surface S and f must vanish on S.

Given five different indices $\alpha, \beta, \gamma, \delta, \epsilon$, we can choose S and f in such a way that, at a given point of S, only the restrictions of $\omega^\alpha, \omega^\beta, \omega^\gamma, \omega^\delta$ do not vanish and only $A_\epsilon f$ is not zero. In this way we obtain the conditions

$$\frac{\partial\lambda_{\alpha\beta\gamma\delta}}{\partial A_\epsilon \Psi^U} = 0, \qquad \frac{\partial\lambda_{\alpha\beta\gamma\delta}}{\partial F^\eta_{\epsilon\zeta}} = 0. \tag{4.24}$$

This means that these expression vanish if all the indices $\alpha, \beta, \gamma, \delta$ are different from ϵ. Therefore we can write, for any value of the indices,

$$\omega^\epsilon \wedge \frac{\partial\lambda}{\partial A_\epsilon \Psi^U} = 0, \qquad \omega^\epsilon \wedge \frac{\partial\lambda}{\partial F^\eta_{\epsilon\zeta}} = 0, \tag{4.25}$$

with no sum over the index ϵ.

Since these condition hold for any choice of the basis in the space $\mathcal{T}$, for any choice of the coefficients ξ_ϵ we have

$$\xi_\theta \xi_\epsilon \omega^\theta \wedge \frac{\partial\lambda}{\partial A_\epsilon \Psi^U} = 0, \qquad \xi_\theta \xi_\epsilon \omega^\theta \wedge \frac{\partial\lambda}{\partial F^\eta_{\epsilon\zeta}} = 0 \tag{4.26}$$

and we obtain the following equations

$$\omega^\theta \wedge \frac{\partial\lambda}{\partial A_\epsilon \Psi^U} + \omega^\epsilon \wedge \frac{\partial\lambda}{\partial A_\theta \Psi^U} = 0, \tag{4.27}$$

$$\omega^\theta \wedge \frac{\partial\lambda}{\partial F^\eta_{\epsilon\zeta}} + \omega^\epsilon \wedge \frac{\partial\lambda}{\partial F^\eta_{\theta\zeta}} = 0. \tag{4.28}$$

We call them the *normal field equations* because they are obtained by considering the derivatives of the fields normal to the integration surface. They have no analog in field theries based on spacetime.

If we apply the interior product operator $i_\epsilon = i(A_\epsilon)$ to these equations, we obtain

$$\frac{\partial\lambda}{\partial A_\epsilon \Psi^U} = \omega^\epsilon \wedge \pi_U, \qquad \pi_U = (n-3)^{-1} i_\epsilon \frac{\partial\lambda}{\partial A_\epsilon \Psi^U}, \tag{4.29}$$

$$\frac{\partial\lambda}{\partial F^{\eta}_{\epsilon\zeta}} = (n-3)^{-1}\omega^{\epsilon}\wedge i_{\theta}\frac{\partial\lambda}{\partial F^{\eta}_{\theta\zeta}}. \tag{4.30}$$

By using the last equation twice, after some calculations, we obtain

$$\frac{\partial\lambda}{\partial F^{\eta}_{\epsilon\zeta}} = -2^{-1}\omega^{\epsilon}\wedge\omega^{\zeta}\wedge\sigma_{\eta}, \qquad \sigma_{\eta} = 2(n-3)^{-1}(n-2)^{-1}i_{\epsilon}i_{\zeta}\frac{\partial\lambda}{F^{\eta}_{\epsilon\zeta}}. \tag{4.31}$$

Now we simplify the action principle by means of the equations (4.29), (4.31) and we derive another set of field equations called the *tangential field equations*. We consider a general choice of $\delta\Psi^U$ and $\delta\omega^\eta$ and we have

$$\begin{aligned}\delta\lambda = \delta\Psi^U\tfrac{\partial\lambda}{\partial\Psi^U} + \delta(A_\alpha\Psi^U)\omega^\alpha\wedge\pi_U + \delta\omega^\alpha\wedge i_\alpha\lambda - 2^{-1}\delta F^{\eta}_{\alpha\beta}\,\omega^\alpha\wedge\omega^\beta\wedge\sigma_\eta \\ = \delta\Psi^U\tfrac{\partial\lambda}{\partial\Psi^U} + d\delta\Psi^U\wedge\pi_U - A_\alpha\Psi^U\delta\omega^\alpha\wedge\pi_U \\ +\delta\omega^\alpha\wedge i_\alpha\lambda + d\delta\omega^\eta\wedge\sigma_\eta + F^{\eta}_{\alpha\beta}\,\delta\omega^\alpha\wedge\omega^\beta\wedge\sigma_\eta.\end{aligned} \tag{4.32}$$

By means of the generalized Stokes theorem, we obtain

$$\begin{aligned}\delta\textstyle\int_S\lambda = \int_{\partial S}(\delta\Psi^U\pi_U + \delta\omega^\alpha\wedge\sigma_\alpha) + \int_S\delta\Psi^U\left(-d\pi_U + \tfrac{\partial\lambda}{\partial\Psi^U}\right) \\ + \textstyle\int_S\delta\omega^\alpha\wedge\left(d\sigma_\alpha + i_\alpha\lambda - A_\alpha\Psi^U\pi_U + F^{\eta}_{\alpha\beta}\,\omega^\beta\wedge\sigma_\eta\right).\end{aligned} \tag{4.33}$$

If $\delta\Psi^U$ and $\delta\omega^\alpha$ vanish on ∂S, the first integral vanishes and, since these quantities are arbitrary at the internal points of S, from the action principle we obtain the following tangential field equations

$$d\pi_U = \frac{\partial\lambda}{\partial\Psi^U}, \tag{4.34}$$

$$d\sigma_\alpha = -\tau_\alpha, \tag{4.35}$$

where

$$\tau_\alpha = -A_\alpha\Psi^U\pi_U - L(A_\alpha)\omega^\eta\wedge\sigma_\eta + i_\alpha\lambda. \tag{4.36}$$

We have introduced the Lie derivative

$$L(A_\alpha)\omega^\eta = -F^{\eta}_{\alpha\beta}\,\omega^\beta. \tag{4.37}$$

It follows that

$$d\tau_\alpha = 0, \tag{4.38}$$

namely the forms τ_α describe quantities that are conserved as a consequance of a Gauss law.

4.3 Noether's theorem

In order to discuss the relation between symmetry and conservation laws, we start from eq. (4.33). We do not assume that the variations of the fields vanish on ∂S, but we take into account the field equations, so that the last two integrals vanish. We obtain

$$\int_{\partial S} \theta = \int_S \delta\lambda, \qquad \theta = \delta\Psi^U \pi_U + \delta\omega^\alpha \wedge \sigma_\alpha. \tag{4.39}$$

Since S is arbitrary, we have

$$d\theta = \delta\lambda \tag{4.40}$$

and if λ is invariant, namely $\delta\lambda = 0$, the 3-form θ is conserved, namely $d\theta = 0$. This equation can be considered as a formulation of the Noether theorem.

We consider two different kinds of infinitesimal symmetry transformations. If we use eq. (2.11), that describes a symmetry property of the Lagrangian, for any element κ of the Lie algebra of the symmetry group $\mathcal{F}$, we obtain the conserved quantity

$$\begin{gathered} \theta(\kappa) = \theta^G(\kappa) + \theta^M(\kappa), \\ \theta^G(\kappa) = C^\alpha{}_\beta(\kappa)\omega^\beta \wedge \sigma_\alpha, \qquad \theta^M(\kappa) = S^U{}_V(\kappa)\Psi^V \pi_U. \end{gathered} \tag{4.41}$$

If we consider an infinitesimal diffeomorphism generated by the infinitesimal vector field B, we have, since λ, does not depend on s explicitly,

$$\begin{gathered} \delta\Psi^U = B\Psi^U, \qquad \delta\omega^\alpha = L(B)\omega^\alpha, \\ \delta\lambda = L(B)\lambda = i(B)d\lambda + di(B)\lambda. \end{gathered} \tag{4.42}$$

If we put

$$\tau(B) = \theta - i(B)\lambda = B\Psi^U \pi_U + L(B)\omega^\alpha \wedge \sigma_\alpha - i(B)\lambda, \tag{4.43}$$

from eq. (4.40) we obtain

$$d\tau(B) = i(B)d\lambda. \tag{4.44}$$

For $B = -A_\alpha$, we have $\tau(-A_\alpha) = \tau_\alpha$ and these quantities are interpreted as the density and the current density of (10+n)-momentum with respect to the moving frame s. We have introduced a minus sign because we are considering *active transformations* of the fields, defined, for scalar fields, by

$$s \to s', \qquad \Psi(s) \to \Psi'(s), \qquad \Psi'(s') = \Psi(s). \tag{4.45}$$

One has to remember that the energy density is described by $\tau^0 = -\tau_0$. In fact, even in elementary mechanics, the Hamiltonian is the generator of the passive time translations.

From eq. (4.38) we obtain

$$i_\alpha d\lambda = -d\tau_\alpha = 0 \tag{4.46}$$

and therefore

$$d\lambda = 0. \tag{4.47}$$

This formula, that follows from the field equations and the invariance of the Lagrangian under diffeomorphisms, implies that the action integral depends only on ∂S and not on the details of S. It is trivially satisfied in a theory based on the 4-dimensional spacetime.

In conclusion, for any choice of the vector field B, as a consequence of the invariance under diffeomorphisms, we have the conservation law

$$d\tau(B) = 0. \tag{4.48}$$

Note that the conservation laws can be easily derived from the field equations. We have derived them from the action integral for pedagogical reasons and to make more clear their connection with the symmetry properties.

In order to show that eq. (4.48) follows from a generalized Gauss law, we introduce the 2-form

$$\sigma(B) = -b^\alpha \sigma_\alpha, \qquad B = b^\alpha A_\alpha \tag{4.49}$$

and, by means of the formula

$$L(B)\omega^\alpha = i(B)d\omega^\alpha + di(B)\omega^\alpha = b^\beta L(A_\beta)\omega^\alpha + db^\alpha, \tag{4.50}$$

we see that

$$d\sigma(B) = -db^\alpha \wedge \sigma_\alpha + b^\alpha \tau_\alpha = -\tau(B). \tag{4.51}$$

In the following Sections, we consider Lagrangian forms invariant with respect to the infinitesimal Lorentz transformations (2.13) and from eq. (4.41) we obtain the conserved quantities

$$\theta_{[ik]} = \hat{F}^\alpha_{[ik]\beta}\omega^\beta \wedge \sigma_\alpha + \Sigma_{[ik]}{}^U{}_V \Psi^V \pi_U. \tag{4.52}$$

Their conservation does not depend on the geometry of $\mathcal{S}$. If $\mathcal{S}$ is a bundle of frames, as in Chapter 1, we have

$$\tau_{[ik]} = \theta_{[ik]} + i_{[ik]}\lambda. \tag{4.53}$$

Note that

$$di_{[ik]}\lambda = L(A_{[ik]})\lambda - i_{[ik]}d\lambda = 0 \tag{4.54}$$

and the three forms that appear in eq. (4.53) are all conserved.

4.4 Minimal coupling and the balance equations

In many interesting cases, there is a *minimal coupling* between geometry and matter, namely a decomposition of the kind

$$\lambda = \lambda^G + \lambda^M, \tag{4.55}$$

where λ^G describes the geometry, namely the gravitation and other gauge fields and does not contain Ψ and its derivatives, while λ^M describes the matter and does not contain the structure coefficients or the two-forms $d\omega^\alpha$. We shall see in Section 5.1 that there are terms that depend only on ω^α and can be attributed arbitrarily to λ^G or to λ^M.

The conserved quantities can be decomposed in a similar way:

$$\tau(B) = \tau^G(B) + \tau^M(B), \tag{4.56}$$

$$\tau^G(B) = L(B)\omega^\alpha \wedge \sigma_\alpha - i(B)\lambda^G, \tag{4.57}$$

$$\tau^M(B) = B\Psi^U \pi_U - i(B)\lambda^M \tag{4.58}$$

and the tangential field equation (4.35) can be written in the form

$$d\sigma_\alpha + F^\eta_{\alpha\beta}\,\omega^\beta \wedge \sigma_\eta + i_\alpha\lambda^G = -\tau^M_\alpha, \tag{4.59}$$

$$\tau^M_\alpha = \tau^M(-A_\alpha) = -A_\alpha\Psi^U\pi_U + i_\alpha\lambda^M. \tag{4.60}$$

In general $\tau^M(B)$ is not conserved separately and its conservation law has to be replaced by a *balance equation* that takes into account the forces exerted by the geometric fields. By means of the same procedure used to derive eq. (4.32) and by taking the tangential field equations into account, we obtain

$$\delta\lambda^M = d(\delta\Psi^U \pi_U) + \delta\omega^\alpha \wedge \tau_\alpha^M. \tag{4.61}$$

By introducing the variations (4.42), we have

$$i(B)d\lambda^M + di(B)\lambda^M = L(B)\lambda^M = d(B\Psi^U \pi_U) + L(B)\omega^\alpha \wedge \tau_\alpha^M \tag{4.62}$$

and finally the balance equation

$$d\tau^M(B) = -L(B)\omega^\alpha \wedge \tau_\alpha^M + i(B)d\lambda^M. \tag{4.63}$$

In many cases the last term vanishes. In particular, if λ^M is a homogeneous function of degree $m \neq 0$ of Ψ and its derivatives, we can put

$$\delta\omega^\alpha = 0, \qquad \delta\Psi^U = \zeta\Psi^U, \qquad \delta A_\alpha \Psi^U = \zeta A_\alpha \Psi^U, \qquad \delta\lambda^M = m\zeta\lambda^M \tag{4.64}$$

and from eq. (4.61) we obtain

$$m\lambda^M = d(\Psi^U \pi_U), \qquad d\lambda^M = 0. \tag{4.65}$$

If $\mathcal{S}$ is the bundle of frames of the Minkowski spacetime, and we consider the isomorphism $g \to \hat{s}g$ between $\mathcal{S}$ and the Poincaré group, the vector fields A_α^L, defined by eq. (1.66), commute with all the fundamental vector fields A_α and we have $L(A_\alpha)\omega^\beta = 0$. It follows that the quantities

$$\tau^M(-A_\alpha^L) = D^\beta{}_\alpha(g^{-1})\tau_\beta^M \tag{4.66}$$

are conserved. They describe the energy-momentum and the relativistic total angular momentum of matter with respect to the fixed frame $\hat{s}$. No symmetry property of the Lagrangian form is required for these conservation laws.

For $B = -A_\alpha$ we have

$$d\tau_\alpha^M = -F_{\alpha\beta}^\gamma \omega^\beta \wedge \tau_\gamma^M. \tag{4.67}$$

We shall use this formula in Chapter 8 in order to describe the motion of a test particle in external geometric fields. The differential 3-forms τ_α^M describe the density and the flow of the (10+n)-momentum of matter (see Section 1.9). The balance equations show that the change of a component of the (10+n)-momentum is given by the product of some other components of (10+n)-momentum and some geometric fields. For instance, a change of momentum is given by the product of the electric charge and the electric field. All the relations of this kind are contained in a compact way in eq. (4.67).

We see that the structure constants of the Poincaré group play a role similar to the role played by the electromagnetic field. This remark gives another support to the idea that all the structure coefficient should have a dynamical nature.

If we consider spatially localized quantities of the kind (4.4), or, more in general

$$\tau_\alpha^M = T_\alpha^k \eta_k, \tag{4.68}$$

we assume that $\mathcal{S}$ has the structure of principal bundle as in Chapter 1, and we use the formula

$$d\eta_i = -F_{ik}^k \eta + 2^{-1} F_{[kl]i}^j \omega^{[kl]} \wedge \eta_j, \tag{4.69}$$

from eq. (4.67) we obtain the Lorentz and gauge transformation properties

$$A_{[ik]} T_\alpha^j = F_{[ik]\alpha}^\beta T_\beta^j - F_{[ik]l}^j T_\alpha^l, \qquad A_a T_\alpha^j = F_{a\alpha}^\beta T_\beta^j \tag{4.70}$$

and the balance equations

$$A_i T_\alpha^i - F_{ik}^k T_\alpha^i = F_{i\alpha}^\beta T_\beta^i. \tag{4.71}$$

In particular we have the equation

$$A_j T_{[ik]}^j - F_{jk}^k T_{[ik]}^j = g_{ij} T_k^j - g_{kj} T_i^j, \tag{4.72}$$

showing that, even in the absence of external fields, the spin density is not conserved, unless the energy momentum tensor T_{ik} is symmetric.

4.5 Pre-symplectic formalism and double differential forms

Many quantization procedures start from some kind of canonical formalism. A covariant canonical formalism on a group manifold has been developed in refs. [89, 90]. A covariant symplectic approach to geometric field theories proposed in ref. [91] can be adapted very naturally to the theories described in the present notes and we shall follow many of its ideas, that will play an important role in the following. The symplectic structure of the phase space is the starting point of geometric quantization [67, 68]. Though we have no reasonable hope to carry out the whole quantization of the classical theories we are considering, the general concepts given in the present Section may suggest some useful ideas.

It is useful to start from the analogy with a mechanical system with d degrees of freedom [92–94]. The *phase space* Γ can be interpeted as the *space of motions*, namely the space of the solutions of the equations of motion or of the corresponding initial conditions at a given time t. We indicate by q^χ, $\chi = 1, \ldots, d$ a local system of Lagrangian coordinates, by $\dot{q}^\chi$ the corresponding velocities and we define, as usual, the canonical momenta

$$p_\chi(t) = \frac{\partial L}{\partial \dot{q}^\chi}, \tag{4.73}$$

where $L(q^\chi, \dot{q}^\chi)$ is the Lagrange function. The (closed, nondegenerate) *symplectic form* is given by

$$\Omega(t) = \hat{d}q^\chi(t) \wedge \hat{d}p_\chi(t), \qquad \hat{d}\Omega(t) = 0. \tag{4.74}$$

We use the symbol $\hat{d}$ to denote the exterior derivative of differential forms defined on Γ, in order to avoid confusion with the exterior derivative of differential forms defined on $\mathcal{S}_n$, that we continue to indicate by d. We also introduce the notation $\hat{\imath}(X)$ and $\hat{L}(X)$ for the inner product and the Lie derivative acting on the differential forms in Γ. For the exterior product, we use in both cases the symbol $\wedge$.

In the following it is useful to consider *double differential forms*, a concept treated with some detail in ref. [95]. A (u, v)-double form defined, for instance, on $\Gamma \times \mathcal{S}_n$ is given, at a given point of this manifold, by a multilinear form depending (antisymmetrically) on u vectors of the tangent space of Γ and v vectors of the tangent space of $\mathcal{S}_n$. The manifold $\mathcal{S}_n$ can be replaced by the manifold that describes the geometry in other theories, for instance the spacetime $\mathcal{M}$ or, in the simple mechanical system we are now considering, the time axis $\mathbf{R}$.

The time evolution of the system is described by a one-parameter group of diffeomorphisms of Γ generated by a vector field X. The dynamical variables are functions defined on $\Gamma \times \mathbf{R}$, that we can also consider as $(0, 0)$-double differential forms. Their time derivative is given by

$$\frac{d}{dt} f = \dot{f} = Xf = \hat{\imath}(X)\hat{d}f \tag{4.75}$$

and the canonical equations of motion can be written in the form

$$\hat{\imath}(X)\Omega = \hat{d}H, \tag{4.76}$$

where $H(q^\chi, p_\chi)$ is the Hamiltonian function. It follows that

$$\frac{d}{dt}\Omega = \hat{L}(X)\Omega = \hat{d}\hat{\imath}(X)\Omega = \hat{d}\hat{d}H = 0, \tag{4.77}$$

namely the symplectic form Ω does not depend on time. This is the property that justifies its definition.

We can consider Ω as a double $(2, 0)$-form on the manifold $\Gamma \times \mathbf{R}$. More explicitly, it is at the same time a 2-form on Γ and a 0-form (namely a function) on $\mathbf{R}$. The fact that it does not depend on t can be written as $d\Omega = 0$, while $\hat{d}\Omega = 0$ means that, for any value of t, it is a closed form on Γ.

If we consider a field theory, the space Γ is infinite-dimensional and the mathematics of differential forms [27] becomes rather delicate and also ambiguous, because one has to choose a norm, or at least a topology, in the tangent spaces. We shall not enter into these details and our proposal is admittedly deprived of any mathematical rigour. We hope that, if necessary, it will be possible to transform it into a mathematically acceptable treatment.

Following the analogy with a mechanical system we treat the quantities ω^α and Ψ^U as "Lagrangian coordinates" and the quantities $F^\gamma_{\alpha\beta}$ and $A_\alpha\Psi^U$ as the "velocities". A comparison between the normal field equations written in the form (4.29), (4.31) and eq. (4.73) suggests that the forms σ_α and π_a should be considers as the "canonical momenta" of the theory.

A natural generalization of eq. (4.74) is

$$\Omega(\Sigma) = \int_\Sigma \Omega, \qquad \Omega = \hat{d}\omega^\alpha \wedge \hat{d}\sigma_\alpha + \hat{d}\Psi^U \wedge \hat{d}\pi_U. \tag{4.78}$$

Under the integral sign, we have the $(2, 3)$-double form Ω, that we call the *symplectic double form* and represents, in some sense, a "density" of symplectic form. The 3-dimensional submanifold Σ of $\mathcal{S}_n$ should be chosen with the same criteria used in Section 4.1 to define the global conserved quantities, for instance the electric charge, starting from the corresponding closed 3-forms that describe their "densities".

The choice of Σ is a difficult problem, but fortunately we can prove that, as a consequence of the field equations, we have $d\Omega = 0$. As in the mechanical case, this is the crucial property of Ω. It follows that $\Omega(\Sigma)$ is not affected by deformations of Σ that modify only a compact subset and not its boundary $\partial\Sigma$. However, Σ should extend to infinity and problems of convergence may arise. Moreover, $\Omega(\Sigma)$ could depend on the asymptotic behavior of Σ.

It has been remarked, in the framework of quantum field theory [96, 97], that globally defined quantities are not really observable and that a field theory should be described in terms of local algebras of observables concerning compact regions of spacetime. Perhaps the form $\Omega(\Sigma)$ with sufficiently large but compact Σ could be sufficient for the construction, by means of some quantization procedure, a local algebra of observables. A further development of this idea is completely outside the purposes of the present notes. In the following we do not specify the choice of Σ, since many local results can be obtained by considering the symplectic double form Ω, without any reference to the symplectic form $\Omega(\Sigma)$.

The infinitesimal variations indicated by δ in the preceding Sections can be described by an infinitesimal vector field X defined on Γ. If f is a dynamical variable, described as a $(0, v)$-double form, its infinitesiamal variation previously indicated by δf, in the present Section should be indicated by $Xf = \hat{\imath}(X)\hat{d}f$.

The 3-form θ defined by eq. (4.39) depends on a vector field X that describes the variations indicated by the symbol δ and it is more correctly interpreted as a (1, 3)double form defined by

$$\theta = \hat{d}\Psi^U \wedge \pi_U + \hat{d}\omega^\alpha \wedge \sigma_\alpha \tag{4.79}$$

and eq. (4.40) gives

$$\hat{\imath}(X)d\theta = \hat{\imath}(X)\hat{d}\lambda. \tag{4.80}$$

Since X is arbitrary, we obtain the important relation between $(1, 4)$-double forms

$$d\theta = \hat{d}\lambda, \tag{4.81}$$

that can be considered asd a formulation of Noether's theorem in terms of double forms. In conclusion, we have

$$\Omega = -\hat{d}\theta, \qquad d\Omega = -d\hat{d}\theta = -\hat{d}\hat{d}\lambda = 0, \tag{4.82}$$

as it was announced above. We also have

$$\Omega(\Sigma) = -\hat{d}\theta(\Sigma), \qquad \theta(\Sigma) = \int_\Sigma \theta \tag{4.83}$$

and we see that, in the circumstances we are considering, the symplectic form $\Omega(\Sigma)$, besides being a closed form, is also exact.

Many interesting theories are described by a degenerate Lagrangian and one can eliminate (locally) the "velocities" $F^{\gamma}_{\alpha\beta}$, $A_\alpha \Psi^U$ from the the normal equations (4.29) and (4.31), obtaining *primary Lagrangian constraints* that involve only the "Lagrangian coordinates" ω^α, Ψ^U and the "canonical momenta" σ_α and π_U. In some cases, from the other field equations one also obtains secondary constraints. As a consequence, the states of the system are described by the points of the submanifold $\Gamma' \subset \Gamma$ defined by all the constraint equations. A more detailed dicussion of primary and secondary constraints can be found, for instance, in ref. [98]. It may be useful to remark that the submanifold Γ' is not necessarily defined by a set of global constraint equations. It may be necessary to use different constraint equations in a neighborhoods of differnt points of Γ'.

The symplectic formalism allows a treatment of the constraints considerably simpler than the better known approach based on the Poisson and Dirac brackets [98]. If the restriction $\Omega'(\Sigma)$ of the form $\Omega(\Sigma)$ to the submanifold Γ' is nondegenerate, Γ' is a symplectic manifold to be identified with the phase space of the system. The corresponding Poisson brackets (different from the Poisson brackets of Γ) are called the *Dirac brackets*.

However, in the most interesting cases the 2-form $\Omega'(\Sigma)$ is degenerate. In this case, the space Γ' is not a symplectic space, but a *pre-symplectic space* and there are vector fields X on Γ' that satisfy the condition

$$\hat{i}(X)\Omega'(\Sigma) = 0. \tag{4.84}$$

These vector fields are interpreted as the generators of gauge transformations. As a consequence of the property

$$\hat{d}\Omega'(\Sigma) = 0, \tag{4.85}$$

they define an integrable distribution of subspaces in the tangent spaces of Γ' and one can apply Frobenius' theorem, as we have done in Section 2.3 (if there is a version of this theorem valid in the infinite-dimensional manifolds we are considering!). If the set Γ'' of the leaves has a manifold structure, one can define on it a symplectic (nondegenerate) form $\Omega''(\Sigma)$ and one obtains in this way the true phase space of the theory.

Alternatively (under suitable conditions), one can introduce, besides the Lagrangian constraints, other *gauge fixing constraints* that define a submanifold $\Gamma'' \subset \Gamma' \subset \Gamma$ that itersects all the leaves of Γ' at only one point and is clearly equivalent to the set of the leaves introduced above. In this case, one can identify $\Omega''(\Sigma)$ with the restriction of $\Omega(\Sigma)$ to Γ''. We can also consider the restriction $\theta''(\Sigma)$ of the 1-form $\theta(\Sigma)$ to Γ'' and we have

$$\Omega''(\Sigma) = -\hat{d}\theta''(\Sigma), \tag{4.86}$$

namely $\Omega''(\Sigma)$ is exact. It is not necessary to assume that Γ'' is defined globally by a set of constraint equations.

However, it is not always possible to find a submanifold Γ'' with the required properties, as one can see from simple finite-dimensional examples. Then the closed form $\Omega''(\Sigma)$ does not need to be exact and may have nontrivial topological (cohomological) properties that can give rise to obstructions to the quantization procedure [67, 68] unless the Planck constant $\hbar$ takes some special values, as we have shortly discussed in Section 2.4. It is for this reason that we try to give some attention to the topological properties of the phase space.

We have already remarked in Section 2.5 that the theories we are considering have gauge transformations corresponding to the diffeomorphisms of $\mathcal{S}_n$. If the vector field B in $\mathcal{S}_n$ generates infinitesimal diffeomorphisms of this manifold, the vector field X on Γ that generates the corresponding gauge transformations is defined by

$$Xf = \hat{i}(X)\hat{d}f = \hat{L}(X)f = L(B)f, \tag{4.87}$$

where f is an arbitrary dynamical variable, namely a $(0, v)$-double form. Since the Lie derivatives are derivations of the algebra of the differential forms, this formula can be extended to an arbitrary (u, v)-double form f, namely we have in general

$$\hat{L}(X)f = L(B)f. \tag{4.88}$$

If there are Lagrangian constraints, the vector field X is tangent to Γ' and it can be considered as a vector field on this submanifold.

By means of the formulas given above and eq. (4.43) and (4.51), we can write

$$\begin{aligned}
\hat{i}(X)\Omega &= -\hat{i}(X)\hat{d}\theta = -\hat{L}(X)\theta + \hat{d}\hat{i}(X)\theta \\
&= -L(B)\theta + \hat{d}(\hat{L}(X)\Psi^U\pi_U + \hat{L}(X)\omega^\alpha \wedge \sigma_\alpha) \\
&= -i(B)d\theta - di(B)\theta + \hat{d}(L(B)\Psi^U\pi_U + L(B)\omega^\alpha \wedge \sigma_\alpha) \\
&= \hat{d}\tau(B) - di(B)\theta = -d(\hat{d}\sigma(B) + i(B)\theta).
\end{aligned} \tag{4.89}$$

It follows that

$$\hat{i}(X)\Omega(\Sigma) = -\int_{\partial\Sigma}(\hat{d}\sigma(B) + i(B)\hat{\theta}) \tag{4.90}$$

and, if B has a compact support that does not intersect the boundary $\partial\Sigma$, this expression vanishes as it is expected if X generates a gauge transformation.

If X generates a symmetry transformation of the kind (2.11), all the quantities that have only contracted indices are invariant and in particular we have

$$\hat{L}(X)\lambda = 0, \qquad \hat{L}(X)\theta = 0, \qquad \hat{L}(X)\Omega = 0. \tag{4.91}$$

It follows that

$$\hat{i}(X)\Omega = -\hat{i}(X)\hat{d}\theta = \hat{d}\hat{i}(X)\theta, \tag{4.92}$$

$$\hat{i}(X)\Omega(\Sigma) = \hat{d}\hat{i}(X)\theta(\Sigma). \tag{4.93}$$

We see that $\hat{i}(X)\theta(\Sigma)$ is the generator of the infinitesimal symmetry transformation in the same sense as the Hamiltonian is the generator of the time evolution in eq. (4.76). Note that this quantity is just the one that is conserved according to the Noether theorem of Section 4.3.

As a further application of the double forms, we consider the addition to the Lagrangian form of an exact term of the kind $d\mu$, where the 3-form μ depends only on ω^α and Ψ^U, namely on the "Lagrangian coordinates", but not on the "velocities". We know that the field equations remain unchanged. We have

$$\lambda \to \lambda + d\mu = \lambda + d\omega^\alpha \wedge i(A_\alpha)\mu + d\Psi^U \wedge \frac{\partial\mu}{\partial\Psi^U}. \tag{4.94}$$

It follows that

$$\sigma_\alpha \to \sigma_\alpha + i(A_\alpha)\mu, \qquad \pi_U \to \pi_U + \frac{\partial\mu}{\partial\Psi^U} \tag{4.95}$$

and

$$\theta \to \theta + \hat{d}\omega^\alpha \wedge i(A_\alpha)\mu + \hat{d}\Psi^U \wedge \frac{\partial\mu}{\partial\Psi^U} = \theta + \hat{d}\mu, \qquad \Omega \to \Omega. \tag{4.96}$$

The symplectic double form Ω is not affected by the new term.

If the vector field X defined on Γ generated a symmetry transformation leaving the Lagrangian invariant, for the corresponding conserved quantity θ we have

$$\theta = \hat{i}(X)\theta \to \theta + \hat{i}(X)\hat{d}\mu = \theta + X\mu. \tag{4.97}$$

We see that θ remains unchanged if μ is symmetric.

Chapter 5

Reformulation of some classical field theories

5.1 The Einstein-Cartan theory of gravitation

In the present Chapter we treat some classical field theories in the space $\mathcal{S}$, which are symmetric with respect to the Lorentz group, do not satisfy the equity principle and do not contain a fundamental length ℓ (see Section 2.4). They allow an alternative formulation based on the spacetime $\mathcal{M}$, but some fields may acquire a geometric meaning that is not present in the spacetime formulation. A careful study of these theories is necessary for the search of new theories with a higher symmetry group, that is the object of Chapter 7.

We start with the Einstein-Cartan theory of gravitation [37,38], that generalizes Einstein's theory by allowing nonvanishing values of the torsion and an influence of the spin density on the geometry. It may be considered as a step towards a theory in which all the structure coefficients have a physical relevance. However, in this theory the torsion is not a really new dynamical variable, since it is given as an algebraic function of the spin density. The physical consequences of the Einstein-Cartan theory and of General Relativity cannot be distinguished by means of the presently available experimental techniques. In a first approach, we disregard the presence of electromagnetic fields and of other internal gauge fields, namely we consider a 10-dimensional space $\mathcal{S}$.

A Lagrangian form that describes the Einstein-Cartan theory in the space

$\mathcal{S}$ has been proposed in ref. [3]. A simpler and more elegant Lagrangian has been proposed independently by Ne'eman and Regge in refs. [51, 52]. In the following we consider a slight generalization of the Ne'eman-Regge Lagrangian form and we decompose it into a hard part λ^H, that depends linearly on the structure coefficients through the exterior derivatives $d\omega^\alpha$, and an additional part λ^A, which does not contain them and could equally well be considered as a part of the matter Lagrangian λ^M. For instance, one can include a cosmological constant k_4 in the potential (5.61) of the Higgs scalar field.

We put $\lambda^G = \lambda^H + \lambda^A$ with

$$\begin{gathered}\lambda^H = (k - k_1)\epsilon_{ikjl}\, d\omega^{[ik]} \wedge \omega^j \wedge \omega^l \\ +2k_1\epsilon_{ikjl}\, d\omega^i \wedge \omega^{[kj]} \wedge \omega^l - k_2\epsilon_{ikjn}g_{lm} d\omega^{[ik]} \wedge \omega^{[jl]} \wedge \omega^{[mn]}, \end{gathered} \tag{5.1}$$

$$\lambda^A = k_3\epsilon_{ikjl}\, g_{mn}\, \omega^{[im]} \wedge \omega^{[nk]} \wedge \omega^j \wedge \omega^l - k_4\eta, \tag{5.2}$$

where η is defined in Section 0.3, $k = k_3 = (32\pi G)^{-1}$ and G is Newton's gravitational constant. In Section 6.1 we need this formula with $k_3 \neq k$.

One can show that these equations give the most general Lorentz invariant 4-form depending on the 1-forms ω^α and at most linearly on the 2-forms $d\omega^\alpha$, with the property that it is odd under space reflection, namely it contains the antisymmetric quantity ϵ_{ikjl}. One can easily see that a 4-form with these properties depending only on the 1-forms $\omega^{[ik]}$ necessarily vanishes.

In refs. [51, 52] the special case with $k_1 = k_2 = k_4 = 0$ was considered. The two terms proportional to k_1 and k_2 are exact forms, since they can be written as

$$-d\left(k_1\epsilon_{ikjl}\, \omega^{[ik]} \wedge \omega^j \wedge \omega^l + 3^{-1}k_2\epsilon_{ikjn}g_{lm}\omega^{[ik]} \wedge \omega^{[jl]} \wedge \omega^{[mn]}\right). \tag{5.3}$$

They do not affect the field equations and the action integral, if $\delta\omega^\alpha = 0$ on the boundary ∂S. However, they influence the definition of the four-momentum and angular momentum of the geometric fields, which are known to be ambiguous in General Relativity [28, 99]. They also play a role in the construction of the theories treated in Section 6.1 and in the Chapter 7. The term proportional to k_4 takes into account a *cosmological constant*, that has raised a considerable interest in the recent years [100, 101].

The normal field equations are automatically satisfied and we have

$$\begin{gathered}\sigma_{[ik]} = 2(k - k_1)\epsilon_{ikjl}\, \omega^j \wedge \omega^l - 2k_2\epsilon_{ikjn}g_{lm}\omega^{[jl]} \wedge \omega^{[mn]}, \\ \sigma_i = 2k_1\epsilon_{ikjl}\, \omega^{[kj]} \wedge \omega^l. \end{gathered} \tag{5.4}$$

These quantities do not depend on the structure coefficients and we can write

$$\lambda^H = d\omega^\alpha \wedge \sigma_\alpha. \tag{5.5}$$

As a consequence, we obtain the following simplified formulas for the gravitational 10-momentum (4.57) and for the geometric tangential field equations (4.59):

$$\tau^G_\alpha = d\omega^\beta \wedge i_\alpha \sigma_\beta + i_\alpha \lambda^A, \tag{5.6}$$

$$d\sigma_\alpha + d\omega^\beta \wedge i_\alpha \sigma_\beta + i_\alpha \lambda^A = -\tau^M_\alpha. \tag{5.7}$$

Even in a flat spacetime, the forms τ^G_α are not spatially localized, since they contain the forms $\omega^{[ik]}$. In a theory in which all the structure coefficients have a dynamical role, this flow of 10-momentum can be considered as the source of the structure constants of the Poincaré group. This is a relevant change of perspective, which can influence the construction of new modified classical theories of gravitation.

The tangential field equations have the explicit form

$$\begin{aligned} &2k\epsilon_{ikjl} F^j_{\alpha\beta} \omega^\alpha \wedge \omega^\beta \wedge \omega^l \\ &-2k_3(\epsilon_{injl} g_{km} - \epsilon_{knjl} g_{im}) \omega^{[mn]} \wedge \omega^j \wedge \omega^l = \tau^M_{[ik]}, \end{aligned} \tag{5.8}$$

$$\begin{aligned} &k\epsilon_{ikjl} F^{[kj]}_{\alpha\beta} \omega^\alpha \wedge \omega^\beta \wedge \omega^l \\ &-2k_3 \epsilon_{ijkl} g_{mn} \omega^{[jm]} \wedge \omega^{[nk]} \wedge \omega^l + k_4 \eta_i = \tau^M_i. \end{aligned} \tag{5.9}$$

We call them the *vertical* and the *horizontal* tangential equations. As it was expected, the coefficients k_1 and k_2 have disappeared.

If the 10-momentum of matter has the spatially localized form (4.68), we obtain

$$F^i_{jk} + \delta^i_j F^l_{kl} - \delta^i_k F^l_{jl} = 8\pi G T^i_{[jk]}, \qquad F^i_{[jk]l} = \hat{F}^i_{[jk]l}, \qquad F^i_{[jk][mn]} = 0, \tag{5.10}$$

$$\begin{aligned} -F^{[ik]}_{jk} + 2^{-1}\delta^i_j F^{[lk]}_{lk} &= 8\pi G(T^i_j - k_4 \delta^i_j), \\ F^{[ik]}_{[jl]m} = 0, \qquad F^{[ik]}_{[jl][mn]} &= \hat{F}^{[ik]}_{[jl][mn]}, \end{aligned} \tag{5.11}$$

namely the field equations of the Einstein-Cartan theory, together with the properties of the structure coefficients that are expected when $\mathcal{S}$ is the bundle of the Lorentz frames of a spacetime manifold. We stress that all these properties are consequences of the action principle.

Besides these equations, we have to take into account two Bianchi identities, which are special cases of eq. (1.55). Also the tensor transformation properties of curvature and torsion follow from eq. (1.55), as we have already observed in Section 1.7.

5.2 Internal gauge theories

A Lagrangian form in an extended space $\mathcal{S}$ describing Maxwell's electromagnetism has been proposed in ref. [3] and it has been generalized to noncommutative gauge theories in ref. [54]. In order to give a motivation, in the electromagnetic case one can start from the Maxwell form $\sigma_\bullet$ defined in eq. (4.12), which appears in the Gauss law (4.13). The generalization to an arbitrary *internal gauge theory* is

$$\sigma_a = 2^{-2} G_{ab} F^b_{ik} \epsilon^{ik}{}_{jl}\, \omega^j \wedge \omega^l, \qquad d\sigma_a = -\tau_a, \tag{5.12}$$

where the nondegenerate real symmetric matrix G_{ab} is invariant under the internal gauge group $\mathcal{G}$, namely it has the property

$$\hat{F}^d_{ab} G_{dc} + \hat{F}^d_{ac} G_{bd} = 0. \tag{5.13}$$

Eq. (5.12) can be obtained from the extended geometric Lagrangian form $\lambda^G = \lambda^H + \lambda^A + \lambda^I$ where the new term is given by

$$\begin{aligned}\lambda^I = d\omega^a \wedge \sigma_a + 2^{-3} G_{ab} \hat{F}^a_{cd} F^b_{ik} \epsilon^{ik}{}_{jl}\, \omega^c \wedge \omega^d \wedge \omega^j \wedge \omega^l \\ + 2^{-2} G_{ab} F^a_{ik} F^b_{jl} g^{ij} g^{kl} \eta.\end{aligned} \tag{5.14}$$

In the usual formulation of the Standard Model of the elementary particles, the matrix G_{ab} contains the coupling constants of the theory. Alternatively, one can rescale the vector fields A_a the forms ω^a, the structure constants $\hat{F}^a_{bc}$ and the matrices Σ_a in such a way that $G_{ab} = \delta_{ab}$. Then, the coupling constants are contained in $\hat{F}^a_{bc}$ and in Σ_a. For the Maxwell theory we adopt the standard convention with rationalized units, namely we put $G_{\bullet\bullet} = 1$ and the coupling constant, namely the elementary charge, appears in $\Sigma_\bullet$, namely in the gauge transformation law of the charged fields given by eq. (1.28). If one prefers nonrationalized units, one has to put $G_{\bullet\bullet} = (4\pi)^{-1}$.

The Lagrangian form (5.14) depends only on the structure coefficients $F^a_{\alpha\beta}$ with $a \geq 10$, while these coefficients do not appear in the other parts of the Lagrangian form λ. The partial derivatives of λ with respect to $F^a_{\alpha\beta}$ have a contribution from the structure coefficients implicitly contained in $d\omega^a$ and a contribution from the structure coefficients explicitly present in the Lagrangian, namely we have

$$\frac{\partial \lambda}{\partial F^a_{\epsilon\zeta}} = -2^{-1}\omega^\epsilon \wedge \omega^\zeta \wedge \sigma_a + \left(\frac{\partial \lambda^I}{\partial F^a_{\epsilon\zeta}}\right)_E, \tag{5.15}$$

where the subscript E indicates the second contribution, that, as we see from eq. (5.14), vanishes if $\epsilon > 3$ or $\zeta > 3$. The first contribution disappears from the normal field equation (4.28) and we easily see that, for $\eta = a$, this equation is equivalent to the simpler condition

$$\left(\frac{\partial \lambda^I}{\partial F^a_{ik}}\right)_E = 0. \tag{5.16}$$

More explicitly we have

$$\begin{aligned} &-2^{-3}F^a_{\alpha\beta}\epsilon^{ik}{}_{jl}\,\omega^\alpha \wedge \omega^\beta \wedge \omega^j \wedge \omega^l \\ &+2^{-3}\hat{F}^a_{cd}\epsilon^{ik}{}_{jl}\,\omega^c \wedge \omega^d \wedge \omega^j \wedge \omega^l + 2^{-1}g^{ij}g^{kl}F^a_{jl}\eta = 0 \end{aligned} \tag{5.17}$$

and, considering various values of the indices α and β, we obtain the normal field equations in the form

$$F^a_{bc} = \hat{F}^a_{bc}, \qquad F^a_{\beta\gamma} = -F^a_{\gamma\beta} = 0, \quad \beta \leq 9, \quad \alpha \geq 4. \tag{5.18}$$

If these equations are satisfied, the quantity (5.12) is the same that appears in eq. (4.31) and we have

$$\lambda^I = -2^{-2}G_{ab}F^a_{ik}F^b_{jl}g^{ij}g^{kl}\eta. \tag{5.19}$$

We describe gravitation as in Section 5.1, but in the extended space $\mathcal{S}$. The presence of λ^I affects the tangential field equations (4.59) by adding to the source term τ^M_i the new term

$$\begin{aligned} \tau^I_i &= F^a_{ik}\,\omega^k \wedge \sigma_a + i_i\lambda^I = I^j_i\eta_j, \\ I^j_i &= G_{ab}(F^a_{im}F^b_{ln}g^{mn}g^{lj} - 2^{-2}F^a_{pm}F^b_{qn}g^{mn}g^{pq}\delta^j_i), \end{aligned} \tag{5.20}$$

that represents the energy-momentum of the gauge field. There is no contribution of the gauge field to the spin density, in agreement with the treatment given in ref. [37]. The energy density cannot be negative, namely

$$I^{00} = -I^0_0 = 2^{-1} G_{ab} F^a_{0r} F^b_{0s} g^{rs} \geq 0, \tag{5.21}$$

if the matrix G_{ab} is positive definite.

There also is the additional tangential field equation

$$d\sigma_a + \hat{F}^c_{ab}\, \omega^b \wedge \sigma_c = -\tau^M_a, \tag{5.22}$$

where the right hand side describes the charges of matter. The second term in the left hand side, that describes the charges of the Yang-Mills field, is absent in Maxwell's theory. Note that it is not spatially localized, since it contains ω^b.

If τ^M_a is spatially localized, eq. (5.22) can be written in the more explicit form

$$G_{ab}(g^{ij} g^{kl} A_l F^b_{ik} - g^{ij} g^{kl} F^m_{lm} F^b_{ik} - 2^{-1} g^{im} g^{kn} F^j_{mn} F^b_{ik}) = -T^j_a, \tag{5.23}$$

$$A_a F^b_{ik} = -\hat{F}^b_{ac} F^c_{ik}, \qquad A_{[ik]} F^a_{jl} = \hat{F}^m_{[ik]j} F^a_{ml} + \hat{F}^m_{[ik]l} F^a_{jm}. \tag{5.24}$$

The first formula is the Yang-Mills field equation with torsion corrections. In particular it gives the inhomogeneous *Maxwell equations*. The other formulas give the transformation properties of the gauge field strength with respect to gauge and Lorentz transformations. As we have observed in Section 1.7, these properties also follow from eq. (1.55). From the same general formula we also obtain the equation

$$A_i F^a_{jk} + A_j F^a_{ki} + A_k F^a_{ij} - F^l_{ij} F^a_{lk} - F^l_{jk} F^a_{li} - F^l_{ki} F^a_{lj} = 0, \tag{5.25}$$

that in the electromagnetic theory is just the homogeneous Maxwell equation.

Note that, in this case too, the typical properties of the structure coefficients of a principal fibre bundle are consequences of the action principle.

5.3 Scalar fields

A treatment of matter fields is given in refs. [4, 5, 102], where fields with arbitrary spin are considered. In the present notes we treat only scalar and Dirac fields, which are needed in the Standard Model of elementary particles

[41] to describe the Higgs field and Fermions, namely we put $\lambda^M = \lambda^S + \lambda^D$. We try, as far as it is possible in a classical framework, to take into account some special features of the Standard Model, in particular the chiral character of the internal gauge transformation of Fermions and the mass generation mechanism.

The word "scalar" has two different meanings. From a first point of view, a *scalar field* is invariant under the Lorentz transformations considered as elements of the structural group of a principal fibre bundle. This property will appear as a consequence of the field equations. From another point of view, it is invariant with respect to the Lorentz transformations considered as elements of the symmetry group $\mathcal{F}$ of the Lagrangian and of the field equations.

We describe the real scalar fields of the theory by the one-column matrix Ξ, on which an orthogonal representation of the internal symmetry group operates. A complex scalar field can be decomposed in its real and imaginary parts, that appear as different elements of the one-column matrix Ξ. We indicate by Σ_a^S the real antisymmetric matrices that describe the infinitasimal internal symmetry transformations of this group.

The field equations can be derived from the Lagrangian form [4, 5, 102]

$$\begin{aligned}\lambda^S = &-g^{ik} d\Xi^T A_k \Xi \wedge \eta_i + g^{ik} \Xi^T \Sigma_a^S A_k \Xi \, \omega^a \wedge \eta_i \\ &+ 2^{-1} g^{ik} A_i \Xi^T A_k \Xi \, \eta - V(\Xi)\eta,\end{aligned} \tag{5.26}$$

where $V(\Xi)$ is a function of the scalar fields invariant under the internal symmetry group.

The dericatives of Ξ appear only in λ^S and we have

$$\frac{\partial \lambda}{\partial A_\alpha \Xi} = \omega^\alpha \wedge \pi^S + \left(\frac{\partial \lambda^S}{\partial A_\alpha \Xi} \right)_E, \tag{5.27}$$

where

$$\pi^S = -g^{ik} A_k \Xi \eta_i \tag{5.28}$$

is a one-column matrix. The subscript E indicates a partial derivative that does not take into account the dependence on the quantities $A_\alpha \Xi$ implicitly contained in $d\Xi$. It vanishes if $\alpha > 3$. The term containing π^S disappears from the normal field equation (4.27) and one can easily see that this equation is equivalent to the simpler equation

$$\left(\frac{\partial \lambda^S}{\partial A_i \Xi} \right)_E = 0. \tag{5.29}$$

After some calculations, we obtain the normal field equations in the more explicit form

$$A_{[ik]}\Xi = 0, \qquad A_a\Xi = -\Sigma'_a\Xi. \tag{5.30}$$

They describe the transformation properties of Ξ under the Lorentz and the internal symmetry group. The quantity (5.28) is the same that appears in eq. (4.29) and the tangential equation (4.34) takes the form

$$-g^{ik}A_iA_k\Xi + g^{ik}F^j_{ij}A_k\Xi + \frac{\partial V}{\partial \Xi} + \Psi^T C\frac{\partial M}{\partial \Xi}\Psi = 0. \tag{5.31}$$

Note that λ depends on Ξ through the function V, but also through the mass generating term in the Fermion Lagrangian (5.36) introduced in the following Section.

If we use the normal field equations, the Lagrangian (5.26) takes the form

$$\lambda^S = -(2^{-1}g^{ik}A_i\Xi^T A_k\Xi + V(\Xi))\eta. \tag{5.32}$$

From eq. (4.60), we see that the contribution of the scalar field to the spin density vanishes and the contribution to the energy-momentum and to the charges is given by

$$T'^i_j = g^{ik}A_j\Xi^T A_k\Xi - (2^{-1}g^{lk}A_l\Xi^T A_k\Xi + V(\Xi))\delta^j_i, \tag{5.33}$$

$$T'^i_a = g^{ik}\Xi^T\Sigma'_a A_k\Xi. \tag{5.34}$$

Note that if $V(\Xi)$ is positive, the contribution to the energy density T'^{00} is positive too.

5.4 Spinor fields

As in the case of scalar fields, the spinor character of a field may have two different meanings, namely a transformation property of the kind (1.6), (1.14) with respect to the structural group of a fibre bundle or a transformation property of the kind (2.10), (2.11) with respect to the symmetry group $\mathcal{F}$ of the Lagrangian and of the field equations. In the present Chapter, disregarding internal symmetries, both the groups are $SL(2, \mathbf{C})$, but in the following developments the distinction becomes more relevant. If $\mathcal{S}$ is not a fibre bundle, a structural group may not exist. On the other hand, the geometric symmetry group of the Lagrangian may be larger than $SL(2, \mathbf{C})$.

We use the Majorana representation in which the gamma matrices are real (see Section 3.3) and we consider a set of Majorana fields Ψ (namely real Dirac fields) with a spinor index and other indices describing other degrees of freedom, on which a real orthogonal representation S of the internal symmetry group $\mathcal{I}$ operates. The group $\mathcal{I}$ contains the gauge group $\mathcal{G}$, but it may also contain other elements that represent global symmetries. It is irrelevant to distinguish between upper and lower internal indices. All these indices are understood and Ψ, as well as the 3-form π^D are considered as one-column matrices.

We assume that the components of Ψ anticommute, in order to obtain Fermion fields after quantization. We remark that in a classical theory anticommuting fields cannot be used for the construction of observables, since an arbitrary product of these fields has a vanishing square and it cannot be considered as a real or a complex number, even if it contains an even number of Fermionic fields and commutes with all the other fields.

In other words, the classical fields take their values in a Grassmann algebra, as it is discussed with more deatail in Chapter 10. The analogy with quantum theory requires that it is a complex algebra with an involution $\Psi \to \Psi^*$ that we call "complex conjugation". From the property

$$(\Psi_1 \Psi_2)^* = \Psi_2^* \Psi_1^*, \tag{5.35}$$

we see that the product of two real anticommuting fields is imaginary. This is the reason of the factors i that appear in many formulas in the following.

When expressions containing Fermionic fields appear as sources of classical geometric fields, one has to replace them by the averages of the corresponding quantum fields in a suitable state, a procedure that necessarily requires drastic approximations. Note that an expression of the kind $\Psi^T A \Psi$ vanishes if the numeric matrix A is symmetric. This is also true if Ψ is an Hermitian Fermionic free quantum field and normal products are used.

A complex Dirac field $\Psi_{\mathcal{C}}$ can be decomposed into its real and imaginary parts, which are real fields and appear as different elements of the one-column matrix Ψ. As we shall see in Section 5.5, specific theories of elementary particles are more simply formulated in terms of complex Dirac fields, but for our present purposes a formulation in terms of real fields is more convenient, since it permits a more clear distinction between geometric and internal symmetries.

In quantum theory, one has to be careful with the charge superselection rule [59]: given a charged field, its real part when applied to the vacuum creates a superposition of states with different electric charges, that do not exist in nature. A similar problem arises with quark and gluon fields.

In order to derive the field equations, we start from the following Lagrangian form, written in terms of real fields

$$\lambda^D = -id\Psi^T C\gamma^i\Psi \wedge \eta_i + 2^{-2}i\phi\Psi^T C(\gamma^i\Sigma_{[jk]} + \Sigma_{[jk]}\gamma^i)\Psi\omega^{[jk]} \wedge \eta_i + i\Psi^T C\gamma^i\Sigma_a\Psi\omega^a \wedge \eta_i + i\Psi^T CM\Psi\eta, \tag{5.36}$$

where the forms η_i and η are defined in Section 0.3 and the matrices $\Sigma_{[jk]}$ that represent the Lorentz Lie algebra, are defined by eq. (1.17). The matrices Σ_a describe the infinitesinal transformations of the internal gauge group as in eq. (1.25). In the Einstein-Cartan geometry of Section 5.1 we have to put $\phi = 1$ and in the geometry described in Section 6.1 ϕ is defined by eq. (6.1). In the second case, we have not a minimal coupling of the kind discussed in Section 4.4. The real matrix M, that determines the masses of the Fermions, may contain the Higgs fields. The matrices $C\gamma^i\Sigma_a$ and CM may contain γ_5 and must be antisymmetric. M and Σ_a are invariant under $SL(2C)$ and commute with γ_5.

The normal field equations are automatically satisfied and we have

$$\pi^D = -iC\gamma^i\Psi\eta_i. \tag{5.37}$$

The tangential field equations take the form

$$A_{[ik]}\Psi = -\phi\Sigma_{[ik]}\Psi, \tag{5.38}$$

$$A_a\Psi = -\Sigma_a\Psi, \tag{5.39}$$

$$\gamma^i A_i\Psi - 2^{-1}F^k_{ik}\gamma^i\Psi + M\Psi = 0. \tag{5.40}$$

We have used the formula (4.69). Note that the Dirac equation and the transformation properties of the fields are field equations on the same footing.

The eigenvalues of $\hbar M$ are the Fermion masses, possibly after a change of sign. It is important to remember that, given a classical field configuration, in the corresponding quantum state, in the limit $\hbar \to 0$, mass, energy, momentum, charge and spin of a particle are proportional to $\hbar$, while the particle number density is proportional to $\hbar^{-1}$.

The Lagrangian form vanishes as a consequence of the field equations and the energy-momentum, the angular momentum and the charges are described the spatially localizable 3-forms of the kind (4.68) with

$$T^i_k = iA_k\Psi^T C\gamma^i\Psi, \tag{5.41}$$

$$T^i_{[jk]} = i\phi\Psi^T C\Sigma_{[jk]}\gamma^i\Psi = \phi\epsilon^i{}_{jkl}W^l, \tag{5.42}$$

$$T^i_a = i\Psi^T C\Sigma_a\gamma^i\Psi, \tag{5.43}$$

$$\theta^D_{[jk]} = \phi^{-1}\tau^D_{[jk]} = \epsilon^i{}_{jkl}W^l\eta_i. \tag{5.44}$$

Note the following formula that we shall use in Section 6.1

$$\frac{\partial\lambda^D}{\partial\phi} = (2\phi)^{-1}\omega^{[jk]} \wedge \tau^D_{[jk]} = (2\phi)^{-1}T^i_{[jk]}\omega^{[jk]} \wedge \eta_i. \tag{5.45}$$

We have introduced a 6-vector with components

$$W^u = 2^{-1}i\Psi^T\breve{\Theta}^u\Psi, \tag{5.46}$$

$$W^l = 2^{-1}i\Psi^T C\gamma^l\gamma^5\Psi, \quad W^4 = 2^{-1}i\Psi^T C\Psi, \quad W^5 = 2^{-1}i\Psi^T C\gamma^5\Psi. \tag{5.47}$$

All the bilinear forms that are trivial with respect to the internal indices and imply only the geometric (spinor) indices are linear combinations of the quantities W^u, that are invariant with respect to the internal symmetry transformations, including charge conjugation. They are natural candidates to provide sources for the purely geometric (gravitational) fields. We have seen in Section 5.1 that W^i are the sources of torsion in the Einstein-Cartan theory. It is natural to expect that the other components W^u play a similar role in gravitational theories with a higher symmetry group.

The other components of

$$\theta^M_{[uv]} = -i(\Sigma_{[uv]}\Psi)^T C\gamma^i\Psi\eta_i \tag{5.48}$$

do not appear in conservation laws derived by the Noether theorem, until higher symmetries are present. However, they are useful in the following and are given by

$$\theta^M_{[45]} = -W^i\eta_i, \qquad \theta^M_{[k4]} = -W^5\eta_k, \qquad \theta^M_{[k5]} = W^4\eta_k. \tag{5.49}$$

If we apply eq. (4.63) to the infinitesimal transformations of the subgroup $U(1)_5$ generated by Σ_{45} (see eqs. (3.76) and (3.77)), we obtain the formula

$$d\theta^M_{[45]} = -i\Psi^T C\gamma^5 M\Psi\eta, \tag{5.50}$$

that can also be written in the form

$$A_iW^i - F^k_{ik}W^i = i\Psi^T C\gamma^5 M\Psi \tag{5.51}$$

and proven directly starting from the Dirac equation (5.40). We see that in a theory in which all the Fermions are massless, the 3-form $\theta^M_{[45]}$ is conserved. This result takes also into account gravitational and other gauge fields, but is not an application of the Noether theorem, because the gravitational Lagrangian is not symmetric under $U(1)_5$.

5.5 Fermions in the Standard Model of elementary particles

In Chapter 7 we need some more details about the matrices Σ_a and M that appear in the Lagrangian (5.36). They are completely specified in the *Standard Model* of elementary particles [41], that explains with great precision a very large amount of experimental data.

In its detailed formulation it is convenient to use complex Dirac fields that can be decomposed into their real and imaginary parts:

$$\Psi_{\mathcal{C}} = \Psi_{\mathcal{R}} + i\Psi_{\mathcal{I}}, \qquad \Psi^* = \Psi_{\mathcal{R}} - i\Psi_{\mathcal{I}}, \qquad \Psi = \begin{pmatrix} \Psi_{\mathcal{R}} \\ \Psi_{\mathcal{I}} \end{pmatrix}. \tag{5.52}$$

In the rest of this Section we drop the subscript $\mathcal{C}$ and we assume that all the Dirac fields are complex. We indicate by Σ_a and M complex matrices that operate on complex fields. They have half the dimension of the matrices indicated by the same symbols in the preceding Section. Note that not all the real matrices can be considered as complex matrices with half dimension: this possibility is a relevant physical assumption. We also adopt the standard notation

$$\overline{\Psi} = -i\Psi^\dagger C \qquad \Psi^\dagger = \Psi^{*T}. \tag{5.53}$$

It is easy to show that the Lagrangian

$$\begin{aligned}\lambda^D = {} & 2^{-1} d\overline{\Psi}\gamma^i\Psi \wedge \eta_i - 2^{-1}\overline{\Psi}\gamma^i d\Psi \wedge \eta_i \\ & -2^{-2}\phi\overline{\Psi}(\gamma^i\Sigma_{[jk]} + \Sigma_{[jk]}\gamma^i)\Psi\omega^{[jk]} \wedge \eta_i \\ & -\overline{\Psi}\gamma^i\Sigma_a\Psi\omega^a \wedge \eta_i - \overline{\Psi}M\Psi\eta,\end{aligned} \tag{5.54}$$

when written in terms of real fields takes the form (5.36). In a similar way we get

$$T^i_k = -2^{-1}A_k\overline{\Psi}\gamma^i\Psi + 2^{-1}\overline{\Psi}\gamma^i A_k\Psi, \tag{5.55}$$

$$W^l = -2^{-1}\overline{\Psi}\gamma^l\gamma^5\Psi, \quad W^4 = -2^{-1}\overline{\Psi}\Psi, \quad W^5 = -2^{-1}\overline{\Psi}\gamma^5\Psi. \tag{5.56}$$

A Dirac field can also be decomposed into the sum of two (necessarily complex) left and right-handed Weyl fields

$$\Psi^L = 2^{-1}(1 + i\gamma_5)\Psi, \qquad \Psi^R = 2^{-1}(1 - i\gamma_5)\Psi. \tag{5.57}$$

They transform separately under proper orthochronous Lorentz transformations (namely under $SL(2, \mathbf{C})$) and they satisfy separate Dirac equations only for vanishing mass. A massless neutrino is described in the original version of the Standard Model by a left-handed Weyl field that has no right-handed partner. We, however, assume that neutrinos have a small mass.

In order to explain the absence of the parity symmetry in the weak interactions, one assumes that the internal gauge symmetries have a *chiral* character, namely they act in a different way on the left and right-handed Weyl fields. This means that the matrices that represent the infinitesimal transformations of the gauge group $\mathcal{G} = SU(2)_W \times SU(3)_C \times U(1)_Y$ have the form

$$\Sigma_a = 2^{-1}(1 + i\gamma_5)\Sigma_a^L + 2^{-1}(1 - i\gamma_5)\Sigma_a^R, \tag{5.58}$$

where the anti-Hermitian matrices Σ_a^L and Σ_a^R represent two different representations of the Lie algebra of $\mathcal{G}$. Note that $C\gamma^i\Sigma_a$ is anti-Hermitian, as it must be in order to have a real Lagrangian.

In the following few formulas we write explicitly the weak isotopic spin indices $t_3 = \pm 1/2$ on which the 2-dimensional complex representation of $SU(2)_W$ operates and the hypercharge indices y. We remember that the electric charge is given by $y/2 + t_3$. The two components of the Higgs field have $y = 1$ and weak isotopic spin $\pm 1/2$. Note that both the spinors

$$\begin{pmatrix} \Xi_{1/2} \\ \Xi_{-1/2} \end{pmatrix}, \qquad \begin{pmatrix} \Xi^*_{-1/2} \\ -\Xi^*_{1/2} \end{pmatrix}. \tag{5.59}$$

transform according to the same representation of $SU(2)_W$.

The same representation acts on the left-handed Fermion fields, while the right-handed field are invariant under $SU(2)_W$. It follows that the new fields

$$\begin{aligned} \Psi'_{y+1} &= v^{-1}(\Psi^L_{y,1/2}\Xi_{-1/2} - \Psi^L_{y,-1/2}\Xi_{1/2}) + \Psi^R_{y+1}, \\ \Psi'_{y-1} &= v^{-1}(\Psi^L_{y,1/2}\Xi^*_{1/2} + \Psi^L_{y,-1/2}\Xi^*_{-1/2}) + \Psi^R_{y-1} \end{aligned} \tag{5.60}$$

are both invariant under $SU(2)_W$. The hypercharge index y takes the value -1 for leptons and $1/3$ for quarks.

For the potential of the Higgs field we adopt the expression

$$V(\Xi) = \lambda(\Xi^\dagger\Xi - v^2)^2, \tag{5.61}$$

that takes its minimum value for $\Xi^\dagger\Xi = v^2$ After the spontaneous symmetry breaking, the Higgs field takes (for instance) the vacuum expectation value

$$< \Xi_{1/2} >= 0, \qquad < \Xi_{-1/2} >= v = v^* \tag{5.62}$$

and we have, disregarding fluctuations of the Higgs field around its vacuum expectation value, $\Psi \approx \Psi'$. It follows that the Fermions acquire their physical masses if we put

$$\hbar\overline{\Psi}M\Psi = \overline{\Psi}'KmK^\dagger\Psi', \tag{5.63}$$

where m is a diagonal mass matrix and K a unitary matrix that contains the *Cabibbo-Kobayashi-Maskawa (CKM) matrix* that mixes the d, s, b quark fields and possibly another matrix that mixes the neutrino flavours. By means of this equation and eq. (5.60), it is easy to find an explicit expression in terms of the Higgs field Ξ for the matrix M that appears in the Lagrangian. Note that the expression (5.63) is invariant with respect to the internal symmetry group and that the matrix CM is anti-Hermitian, as it is required in order to obtain a real Lagrangian.

It is important to remark that in the Standard Model there is no natural correspondence between the left handed and right handed Weyl fields that permits a natural introduction of Dirac fields. This correspondence is generated by the mass term only after the Higgs field has acquired a vacuum expectation value.

However, in order to introduce higher geometric symmetries, we have to specify the action of $SL(4,\mathbf{R})$ on the Dirac fields and this group contains elements that do not commute with γ^5 and mix Weyl fields with different helicities. In other words, the $SL(4,\mathbf{R})$ transformations do not commute with the internal gauge transformations given by eq. (5.58). In Chapter 7 we solve the problem by modifying this formula.

Chapter 6

Theories with a variable gravitational coupling

6.1 A geometrized scalar-tensor theory of gravitation

A very interesting class of generalizations of Einstein's theory is based on the replacement of the gravitational constant G by a variable scalar field [69–71]. A motivation for this assumption is the explanation of the extremely small value of G when measured in atomic units (*Dirac's large numbers problem*). According to these theories, the value of the scalar field that replaces G is determined by the distribution of matter in the universe, in agreement with the ideas of Mach [103] on the influence of very far celestial bodies on the locally observed phenomena. The best known Lagrangian *scalar-tensor theory* of this kind has been proposed by Brans and Dicke [71] and theories including torsion have been discussed in ref. [104].

We think that it is important to give to this scalar field a geometric interpretation, by writing it as a function of the structure coefficients, in agreement with the idea that all the structure coefficients should have a dynamical relevance. In the discussion of this problem we also clarify some concepts introduced in Chapters 2 and 3 and we introduce some ideas useful for the developments of Chapter 7.

We have seen in Section 5.2 that one can include the coupling constants of a Yang-Mills theory in the structure constants F^c_{ab} of the gauge group. If the coupling constant becomes a variable [105], these structure coefficients acquire a dynamical role. In a similar way, one can try to include the gravitational constant G, or the variable field that replaces it, in the structure constants $F^{[mn]}_{[ik][jl]}$ and $F^{l}_{[ik]j}$ of the Poincaré group. A theory of this kind, equivalent, if spin is neglected, to the Brans-Dicke theory [71], has been discussed in ref. [5]. Here we give a slightly different treatment.

We define the scalar field ϕ by means of the formula

$$\phi = (12)^{-1} g^{kj} F^{i}_{[ik]j}. \tag{6.1}$$

Note that if $F^{i}_{[ik]j} = \hat{F}^{i}_{[ik]j}$, we obtain $\phi = 1$. Since we want to preserve the large amount of physical information contained in Einstein's theory, we consider a minimal modification of the Lagrangian form given by eqs. (5.1) and (5.2). First of all, we replace the constants $k, k_1, \ldots, k_4$ by suitable functions of ϕ indicated by the same symbols. Note that adding a constant to the functions k_1 or k_2 results in adding to the Lagrangian an irrelevant exact form, that leaves the field equations unchanged.

Moreover, we have to add new terms to the Lagrangian, in order to obtain the field equations that determine the field ϕ starting from the matter distribution in the universe and from suitable initial conditions. It seems that these new terms should contain the derivatives $A_i\phi$, but we want to avoid the appearance in the Lagrangian of derivatives of the structure coefficients. We try to use for the same purpose suitable functions of the structure coefficients, increasing in this way their physical role, in agreement with our program.

The structure coefficients that appear in the forms $d\omega^\alpha$ are not sufficient and we have to introduce some other expressions χ_i. After several attempts, one finds that it is convenient to put

$$\chi_i = F^{k}_{ik} \tag{6.2}$$

and to add to the Lagrangian the expression

$$\begin{aligned} \lambda^\chi = k_5 \chi_i \epsilon^i{}_{kjl} d\omega^k \wedge \omega^j \wedge \omega^l + k_5 \phi \chi_i \epsilon^i{}_{kjl} g_{mn} \omega^{[km]} \wedge \omega^n \wedge \omega^j \wedge \omega^l \\ + k_5 \chi_i \chi^i \eta, \qquad k_5 \neq 0, \end{aligned} \tag{6.3}$$

which contributes to both λ^H and λ^A. It may look strange to involve the torsion tensor in the description of the variable gravitational coupling, but we shall see that this procedure works.

The derivatives of λ that appear in the normal field equation (4.28) contain two contributions, one coming from the forms $d\omega^\alpha$ and the other originated by the dependence on the quantities ϕ and χ_i, namely we have

$$\frac{\partial\lambda}{\partial F^\eta_{\epsilon\zeta}} = -2^{-1}\omega^\epsilon \wedge \omega^\zeta \wedge \sigma_\eta + \frac{\partial\lambda}{\partial\phi}\frac{\partial\phi}{\partial F^\eta_{\epsilon\zeta}} + \frac{\partial\lambda}{\partial\chi_i}\frac{\partial\chi_i}{\partial F^\eta_{\epsilon\zeta}}, \tag{6.4}$$

where

$$\begin{aligned}\sigma_{[ik]} &= 2(k-k_1)\epsilon_{ikjl}\,\omega^j \wedge \omega^l - 2k_2\epsilon_{ikjn}g_{lm}\omega^{[jl]} \wedge \omega^{[mn]},\\ \sigma_i &= 2k_1\epsilon_{ikjl}\,\omega^{[kj]} \wedge \omega^l + k_5\chi_k\epsilon^k{}_{ijl}\omega^j \wedge \omega^l.\end{aligned} \tag{6.5}$$

The first contribution satisfies the normal field equation automatically and the second contribution gives the condition

$$\left(\frac{\partial\phi}{\partial F^\eta_{\epsilon\zeta}}\omega^\theta + \frac{\partial\phi}{\partial F^\eta_{\theta\zeta}}\omega^\epsilon\right) \wedge \frac{\partial\lambda}{\partial\phi} + \left(\frac{\partial\chi_i}{\partial F^\eta_{\epsilon\zeta}}\omega^\theta + \frac{\partial\chi_i}{\partial F^\eta_{\theta\zeta}}\omega^\epsilon\right) \wedge \frac{\partial\lambda}{\partial\chi_i} = 0. \tag{6.6}$$

By using the definitions of ϕ and χ_i and choosing the indices $\epsilon, \zeta, \eta, \theta$ in a proper way, we obtain, for all the values of $[jk]$

$$\omega^{[jk]} \wedge \frac{\partial\lambda}{\partial\phi} = 0, \qquad \omega^{[jk]} \wedge \frac{\partial\lambda}{\partial\chi_i} = 0. \tag{6.7}$$

For a given value of $[jk]$, the condition $\omega^{[jk]} \wedge \alpha = 0$ implies that the form α is proportional to $\omega^{[jk]}$. If we require that this condition holds for all the six values of $[ik]$, either α vanishes or it is a form of degree not smaller than 6. In conclusion, we obtain the conditions

$$\frac{\partial\lambda}{\partial\phi} = 0, \tag{6.8}$$

$$\frac{\partial\lambda}{\partial\chi_i} = 0, \tag{6.9}$$

that are equivalent to the normal field equations.

We assume that λ^M does not depend on χ_i and that its dependence on ϕ is given by eq. (5.45). The normal equation (6.9) involves only the additional Lagrangian λ^χ and it is equivalent to the conditions

$$F^l_{[ik]j} = \phi\hat{F}^l_{[ik]j}, \qquad F^m_{[ik][jl]} = 0. \tag{6.10}$$

In order to obtain simpler and more consistent results, we choose the functions k_1, k_2 and k_3 in such a way that

$$k_3' = \phi(k' + k_1'), \qquad k_2' = 0, \qquad k' - k_1' \neq 0, \tag{6.11}$$

where k' indicates the derivative of k with respect to ϕ. By taking into account the formula (5.45), the normal equation (6.8) gives the conditions

$$F^{[ik]}_{[jl][mn]} = \phi \hat{F}^{[ik]}_{[jl][mn]}, \tag{6.12}$$

$$2(k' - k_1')F^{[ik]}_{ik} + k_4' + k_5' \chi_i \chi^i = 0, \tag{6.13}$$

$$4k_1'\phi(F^i_{jk} + \delta^i_j F^l_{kl} - \delta^i_k F^l_{jl}) - 2k_5\phi(\chi_j \delta^i_k - \chi_k \delta^i_j) - 4(k' - k_1')\phi F^{[il]}_{l[jk]} = T^i_{[jk]}. \tag{6.14}$$

From eqs. (6.10) and (6.12) and the special case $J^i_{[jk][mn]l} = 0$ of the generalized Jacobi identity (1.55), we see that

$$A_{[ik]}\phi = 0, \tag{6.15}$$

namely that ϕ is indeed a scalar field.

The eqs. (5.5), (5.6) and (5.7) are still valid and the explicit equations (5.8) and (5.9) contain additional terms proportional to k_5 and to the derivatives of ϕ and χ_i. By performing the calculations, if the 10-momentum of matter has the local form (4.68), from the vertical tangential equation we obtain first of all the condition

$$\begin{aligned} 4k(F^i_{jk} + \delta^i_j F^l_{kl} - \delta^i_k F^l_{jl}) - 2k_5\phi(\chi_j \delta^i_k - \chi_k \delta^i_j) \\ -4(k' - k_1')(A_k\phi\delta^i_j - A_j\phi\delta^i_k) = T^i_{[jk]}, \end{aligned} \tag{6.16}$$

which is compatible with the normal equation (6.14) only if

$$k_1'\phi = k, \tag{6.17}$$

$$A_k\phi\delta^i_j - A_j\phi\delta^i_k = \phi F^{[il]}_{l[jk]}. \tag{6.18}$$

Continuing the analysis of the vertical tangential equation, we see that it is compatible with eq. (6.10) only if we choose

$$k_3 = \phi k. \tag{6.19}$$

If we take into account all the conditions found up to now, from the horizontal tangential equation we obtain

$$\phi F^{[pq]}_{i[jk]} = A_i \phi(\delta^p_j \delta^q_k - \delta^q_j \delta^p_k), \tag{6.20}$$

$$\begin{aligned} 4k(-F^{[ik]}_{jk} + 2^{-1}\delta^i_j F^{[lk]}_{lk}) - 2k'_5(\chi^i A_j \phi - \chi^k A_k \phi \delta^i_j) \\ -2k_5(A_j \chi^i - A_k \chi^k \delta^i_j) + (k_4 - k_5 \chi_k \chi^k)\delta^i_j = T^i_j \end{aligned} \tag{6.21}$$

and eq. (6.18) follows as a consequence.

By using the conditions (6.10), from the generalized Jacobi identity (1.55) we obtain the formula

$$J^i_{[jk]li} = A_{[jk]}\chi_l - F^i_{[jk]l}\chi_i + A_j \phi g_{lk} - A_k \phi g_{lj} + \phi F^{[pq]}_{q[jk]} g_{lp} = 0, \tag{6.22}$$

which combined with eq. (6.18) gives the condition

$$A_{[jk]}\chi_l = F^i_{[jk]l}\chi_i, \tag{6.23}$$

showing that χ_l is a vector field.

Note that, if we put $k_5 = 0$ from eqs. (6.13) and (6.21) we have the condition

$$T^i_i = 4k_4 - 2kk'_4(k' - k'_1), \tag{6.24}$$

which is too restrictive, at least if massive particles are present. This means that the introduction of the term (6.3), proportional to k_5, in the Lagrangian cannot be avoided

Attempts to generalize the above discussed model by introducing anisotropic features of the gravitational field are discussed in ref. [108].

6.2 Macroscopic physical interpretation

From these rather boring calculations we have learned several important lessons. We have shown that some scalar fields, which play a peculiar role in modern physics [106], can be generated in a purely geometric way.

We have also seen that we have to carefully choose the Lagrangian in order to get a consistent set of field equations that have a reasonably wide set of solutions. In fact, since the action principle (4.15) must be satisfied for an arbitrary choice of the integration surface S, one often obtains a too restrictive set of field equations. For instance, the choice of the matter Lagrangian is not independent of the choice of the geometric Lagrangian, since eq. (5.45) is necessary in order to avoid a contradiction between the normal equation (6.14) and the tangential equation (6.16)

Moreover, the functions $k, k_1, \ldots, k_5$ cannot be chosen at will, but they must satify eqs. (6.11), (6.17) and (6.19), which are not field equations, but just limitations to the form of the Lagrangian. According to these conditions, that are not independent, k, k_4 and k_5 can be chosen as arbitrary functions of ϕ, but k_3 is uniquely determined and k_1, k_2 are determined up to an irrelevant additive constant, that adds to the Lagrangian an exact differential form.

We are interested in gravitational theories that do not contain the gravitational constant G of Einstein's theory and we also require that they do not contain any other constant with nontrivial dimension (besides the velocity of light). Then k has to be a power of ϕ and a possible numeric constant factor can be absorbed in the definition of the forms $\omega^{[ik]}$. Taking into account the above mentioned constraints, we put, as in ref. [5],

$$\begin{gathered} k = 2^{-2}\phi^{m-1}, \qquad k_1 = 2^{-2}(m-1)^{-1}\phi^{m-1}, \\ k_2 = 0, \qquad k_3 = 2^{-2}\phi^m, \qquad m \neq 1. \end{gathered} \tag{6.25}$$

For $m = 1$, we have

$$k = 2^{-2}, \qquad k_1 = 2^{-2}\log(\phi/\phi_0), \qquad k_2 = 0 \qquad k_3 = 2^{-2}\phi, \tag{6.26}$$

where ϕ_0 is a dimensional constant that, however, is irrelevant.

Other interesting results follow from a *dimensional analysis*. We indicate by $[L]$ the dimension of time and lenght and by $[M]$ the dimension of mass, energy and momentum. Since the coordinates of the manifold $\mathcal{S}$ are dimensionless, the Lagrangian form has the dimension of an action. We have the dimensional relations

$$[\lambda] = [LM], \qquad [\omega^i] = [L], \qquad [\chi_i] = [L^{-1}]. \tag{6.27}$$

It follows from eqs. (5.1) that

$$[\omega^{[ik]}] = [\phi]^{-1} = [L^{-1/(2-m)}M^{1/(2-m)}], \qquad m \neq 2. \tag{6.28}$$

For $m = 2$ the introduction of a dimensional coupling constant is inavoidable, but this choice is already excluded by the conditions (6.11).

From eqs. (5.2) and (6.3) we also have

$$[k_4] = [L^{-3}M], \qquad [k_5] = [L^{-1}M] \tag{6.29}$$

and we have to put

$$k_5 = -\alpha\phi^{m-2}, \tag{6.30}$$

where α is a dimensionless constant. It is not possible to write k_4, that describes a cosmological term, as a function of ϕ without introducing a dimensional constant.

In order to write the field equations in a more familiar form, it is convenient to introduce the new vector fields

$$\tilde{A}_{[ik]} = \phi^{-1}A_{[ik]}, \qquad \tilde{A}_i = A_i. \tag{6.31}$$

As we have discussed in Section 2.2 and we shall see with more detail in the next Section 6.3, this change of basis in the tangent spaces $T_s\mathcal{S}$ may be operationally unjustified from a microscopic point of view.

We also define the new differential 1-forms

$$\tilde{\omega}^{[ik]} = \phi\omega^{[ik]}, \qquad \tilde{\omega}^i = \omega^i \tag{6.32}$$

and, by computing the commutators and taking into account the field equations (6.10), (6.12) and (6.20), we obtain the new structure coefficients

$$\begin{gathered} \tilde{F}^i_{[ik]j} = \phi^{-1}F^i_{[ik]j} = \hat{F}^i_{[ik]j}, \qquad \tilde{F}^{[mn]}_{[ik][jl]} = \phi^{-1}F^{[mn]}_{[ik][jl]} = \hat{F}^{[mn]}_{[ik][jl]}, \\ \tilde{F}^m_{[ik][jl]} = 0, \qquad \tilde{F}^{[mn]}_{j[ik]} = F^{[mn]}_{j[ik]} - \phi^{-1}A_j\phi(\delta^m_i\delta^n_k - \delta^n_i\delta^m_k) = 0. \\ \tilde{F}^j_{ik} = F^j_{ik}, \qquad \tilde{F}^{[mn]}_{ik} = \phi F^{[mn]}_{ik}. \end{gathered} \tag{6.33}$$

They are compatible with a stucture of principal bundle of the manifold $\mathcal{S}$ and the fields $\tilde{A}_{[ik]}$ are the generators of the structural group namely of the Lorentz group.

We remark that, after the substitution (6.31), the field ϕ disappears from the Fermion Lagrangian (5.36) and we obtain a theory with a minimal coupling of the kind considered in Section 4.4. We also introduce the quantities

$$\tilde{T}^i_{[jk]} = \phi^{-1}T^i_{[jk]}, \tag{6.34}$$

that describe the usual angular momentum, corresponding to the infinitesimal transformations generated by $-\tilde{A}_{[jk]}$.

Expressed in terms of the new variables, the eqs. (6.13), (6.16), (6.21), take the form

$$2^{-1}\tilde{F}^{[ik]}_{ik} + \frac{dk_4}{d\Phi} - \alpha\chi_i\chi^i = 0, \tag{6.35}$$

$$\Phi F^i_{jk} + (1-2\alpha)\Phi(\chi_k\delta^i_j - \chi_j\delta^i_k) - (A_k\Phi\delta^i_j - A_j\Phi\delta^i_k) = \tilde{T}^i_{[jk]}, \tag{6.36}$$

$$\begin{aligned}&\Phi(-\tilde{F}^{[ik]}_{jk} + 2^{-1}\delta^i_j\tilde{F}^{[lk]}_{lk}) + 2\alpha(\chi^i A_j\Phi - \chi^k A_k\Phi\delta^i_j)\\ &+2\alpha\Phi(A_j\chi^i - A_k\chi^k\delta^i_j) + (k_4 + \alpha\Phi\chi_k\chi^k)\delta^i_j = T^i_j,\end{aligned} \tag{6.37}$$

where the field Φ given by

$$\Phi = \phi^{m-2} \tag{6.38}$$

will be identified with the Brans-Dicke scalar field. After the change of basis (6.31), it has lost its geometric character since it cannot be written as a function of the new structure coefficients. The fields χ_i are still given by eq. (6.2). The locally measured gravitational constant is $G = (8\pi\Phi)^{-1}$. In some treatments, the factor 8π is included in the definition of Φ and it appears in the field equations.

Note that the dependence of the various coefficients on ϕ and in particular the exponent m have disappeared from these equations. This means that the choice of the fundamental fields $A_{[ik]}$ is ambiguous in this macroscopic context, as we discuss in the next Section 6.3.

If we consider only Dirac spinning particles, from eq. (5.42) we see that eq. (6.36) is equivalent to the equations

$$F^i_{jk} + 3^{-1}(\chi_k\delta^i_j - \chi_j\delta^i_k) = \Phi^{-1}\epsilon^i{}_{jkl}W^l, \tag{6.39}$$

$$(2-6\alpha)\chi_i - 3\Phi^{-1}A_i\Phi = 0. \tag{6.40}$$

Assuming $\alpha \neq 1/3$, we have

$$F^i_{ki} = \chi_k = -\omega\Phi^{-1}A_k\Phi, \tag{6.41}$$

where

$$\omega = 3(6\alpha-2)^{-1}. \tag{6.42}$$

These equations show that the idea of replacing the derivatives of the scalar field by the vector part of torsion was correct.

We see from eq. (6.35) that, even in the absence of spinning particles, the torsion may not vanish. Since it does not appear in the Brans-Dicke equations, in order to compare them with our formulas, we have to eliminate torsion by changing again the choice of the fundamental vector fields, namely by introducing a torsionless connection, by means of the substitution

$$\breve{A}_k = A_k - 3^{-1}\chi^i \tilde{A}_{[ik]}, \qquad \breve{A}_{[ik]} = \tilde{A}_{[ik]}. \tag{6.43}$$

Note that, since Φ is a scalar field and χ_i is a vector field,

$$\breve{A}_k\Phi = A_k\Phi, \qquad \breve{A}_k\chi_i = A_k\chi_i + 3^{-1}\chi_k\chi_i - 3^{-1}\chi_j\chi^j g_{ik}. \tag{6.44}$$

By computing the Lie brackets of the new fields and using eq. (6.39) with $W^l = 0$, we obtain the new structure coefficients

$$\breve{F}^i_{jk} = 0, \tag{6.45}$$

$$\begin{aligned}\breve{F}^{[il]}_{jk} &= \tilde{F}^{[il]}_{jk} - 3^{-2}\chi_m\chi^m(\delta^i_j\delta^l_k - \delta^i_k\delta^l_j)\\ &-3^{-1}(\breve{A}_j\chi^i\delta^l_k - \breve{A}_k\chi^i\delta^l_j - \breve{A}_j\chi^l\delta^i_k + \breve{A}_k\chi^l\delta^i_j)\\ &+3^{-2}(\chi_j\chi^i\delta^l_k - \chi_k\chi^i\delta^l_j - \chi_j\chi^l\delta^i_k + \chi_k\chi^l\delta^i_j).\end{aligned} \tag{6.46}$$

By means of these formulas and of eq. (6.41), we can write eqs. (6.35) and (6.37) in the form

$$-\breve{F}^{[ik]}_{ik} + 2\omega\Phi^{-1}\breve{A}_j\breve{A}^j\Phi - \omega\Phi^{-2}\breve{A}_j\Phi\breve{A}^j\Phi - 2\frac{dk_4}{d\Phi} = 0, \tag{6.47}$$

$$\begin{aligned}&\Phi(-\breve{F}^{[ij]}_{kj} + 2^{-1}\breve{F}^{[jl]}_{jl}\delta^i_k) - \breve{A}_k\breve{A}^i\Phi + \breve{A}_j\breve{A}^j\Phi\delta^i_k\\ &-\omega\Phi^{-1}(\breve{A}_k\Phi\breve{A}^i\Phi - 2^{-1}\breve{A}_j\Phi\breve{A}^j\Phi\delta^i_k) + k_4\delta^i_k = T^i_k\end{aligned} \tag{6.48}$$

and, as a consequence, also taking eq. (6.41) into account, we obtain

$$(3 + 2\omega)\breve{A}_i\breve{A}^i\Phi = T^i_i - 4k_4 + 2\Phi\frac{dk_4}{d\Phi}. \tag{6.49}$$

These are just the field equations of the Brans-Dicke theory [71], with the addition of a variable cosmological term.

The elimination of torsion by means of a different choice of the fundamental vector fields (namely of the connection) is not completely harmless, since the motion of spinning test particles may be influenced by torsion [37].

A comparison with astronomical measurements in the solar systems gives a rather high lower bound on the dimensionless parameter ω. Recent data from Cassini-Huygens spacecraft give $\omega > 4 \times 10^4$ [109]. This means that $\alpha \approx 1/3$ and the field Φ is approximately constant. Then the limit $\omega \to \infty$ or $\alpha \to 1/3$ of the Brans-Dicke equations is physically very interesting, but it is not trivial [110].

In our geometric approach, however, we can introduce the choice $\alpha = 1/3$ from the begining directly in eq. (6.30), in the Lagrangian (6.3) and in eqs. (6.35), (6.37) and (6.39). From eq. (6.40) we see that $A_k \Phi = 0$, namely $\Phi = (8\pi G)^{-1}$ is constant. From eq. (6.20) and (6.2) we also obtain

$$F^{[il]}_{l[jk]} = 0, \qquad \chi_i = F^{\alpha}_{i\alpha}. \tag{6.50}$$

In this way, we obtain a theory with a constant gravitational coupling, that is not determined by the theory, but by the initial conditions. It also has a *dynamical torsion*, since the quantities χ_i, that represent the 4-vector part of the torsion, are true dynamical variables and their derivatives appear in the field equations. Theories with a dynamical torsion have been proposed by various authors.

If the spin density vanishes, by introducing the new fields $\breve{A}_\alpha$, we obtain the equations

$$\Phi(-\breve{F}^{[ij]}_{kj} + 2^{-1}\breve{F}^{[jl]}_{jl}\delta^i_k) + k_4\delta^i_k = T^i_k \tag{6.51}$$

$$-2\Phi\breve{A}_i\chi^i = T^i_i - 4k_4 + 2\Phi\frac{dk_4}{d\Phi}. \tag{6.52}$$

These equations can also be obtained from the Brans-Dike equations (6.48) and (6.49) by introducing the substitution $\omega\Phi^{-1}\breve{A}_i\Phi \to -\chi_i$ before performing the limit $\omega \to \infty$.

The first formula is just the field equation of general relativity. The second formula determines only partially the vector field χ_i that, since torsion has been eliminated, has lost its geometric meaning and does not give any contribution to the source of the gravitational field. These properties of χ_i are hardly acceptable from the physical point of view. One may suggest that the model in not complete and that new terms have to be added to the Lagrangian. We shall consider again this suggestion in Chapter 7.

6.3 Microscopic considerations and dilatations of $\mathcal{T}$.

As we have discussed if Section 2.2, the choice of the fundamental vector fields that determine the absolute parallelism structure of $\mathcal{S}$ cannot be arbitrary if the minimum time necessary to perform physical operations cannot be neglected. In fact, if we consider the above described theory as a macroscopic approximation of a more complete theory in which the cone $\mathcal{T}^+$ and the constant ℓ play a nontrivial role, the fields A_α used to define the cone have to be unambiguously specified (up to a transformation of $GL(4, \mathbf{R})$). Then also the field ϕ and the exponent m acquire a physical relevance.

The constant ℓ that appears in the definition of $\mathcal{T}^+$ has dimension

$$[\ell] = [\omega^i][\omega^{[ik]}]^{-1} = [L][\phi] = [L^{1+1/(2-m)} M^{-1/(2-m)}] \tag{6.53}$$

and in general it is not a length. The acceleration and the angular velocity of a frame are defined in terms of the fields $\tilde{A}_{[ik]}$ and therefore the maximal acceleration introduced by $\mathcal{T}^+$ is given by $\phi\ell^{-1}$ and may depend on the point.

In a theory with variable couplings, one has to decide which parameters are really constant and which are variable fields. Our point of view (not shared by some authors [107]) is that the velocity of light $c = 1$ and the parameter ℓ should be really constant, because they determine the structure of $\mathcal{T}^+$, namely of the fundamental causal structure of the geometry.

In order to avoid serious problems with the quantization procedure, Planck's constant too has to be really constant. If one wants to avoid the proliferation of fundamental constants (an economy principle), an appealing choice is

$$m = 4, \qquad \ell = \nu\hbar^{1/2}, \tag{6.54}$$

where ν is an adimensional factor, presumably of the order of one. We get in this way a classical theory "prepared" for quantization, and the quantization procedure should determine the possible values of ν. As we have noted in Section 2.4, a similar situation also appears in classical theories with constant gravitational coupling G and a fixed fundamental length ℓ.

In ref. [5] the different choice $m = 1$ has been suggested, starting from considerations based on *Mack's principle*. One assumes that the forms $\omega^{[0r]}$ do not measure directly the acceleration of a frame, but the force per unit of gravitational mass acting on the object associated to the frame. The forms

$\tilde{\omega}^{[0r]}$ introduced above measure the acceleration, namely the force per unit of inertial mass and the ratio ϕ^{-1} between the two forms gives the ratio between the inertial and the gravitational mass. According to Mack's principle, the inertia is determined by the matter present in the universe and is proportional to the field Φ, that satisfies the field equation (6.49) in which the energy-momentum of the cosmic matter provides the source. In conclusion, we have, with suitable normalizations, $\Phi = \phi^{-1}$, namely $m = 1$.

Note that in the model we are considering, assuming that the coefficient k_4 vanishes or is proportional to ϕ^m, the gravitational Lagrangian form is an homogeneous functions of degree m of the structure coefficients. It follows that it has a simple behavior under the dilatations of the vector space $\mathcal{T}$, that we call *total dilatations*, since they imply dilatations of both the vertical and the horizontal subspaces. Note that the cone $\mathcal{T}^+$ is invariant under total dilatations, but not under separate vertical or horizontal dilatations. The infinitesimal total dilatations are given in terms of the infinitesimal parameter ζ by

$$\begin{gathered}\delta A_\alpha = \zeta A_\alpha, \qquad \delta\omega^\alpha = -\zeta\omega^\alpha, \qquad \delta F^\gamma_{\alpha\beta} = \zeta F^\gamma_{\alpha\beta},\\ \delta\phi = \zeta\phi, \qquad \delta\chi_i = \zeta\chi_i,\\ \delta\lambda^H = (m-4)\zeta\lambda^H, \qquad \delta\lambda^A = (m-4)\zeta\lambda^A. \end{gathered} \tag{6.55}$$

It is interesting to investigate the behavior of the other parts of the Lagrangian form under total dilatations. For the Dirac Lagrangian (5.36) we put

$$\begin{gathered}\delta\Psi = 2^{-1}(m-1)\zeta\Psi, \qquad \delta\omega^a = 0, \qquad \delta\Xi = \zeta\Xi,\\ \delta\lambda^D = (m-4)\zeta\lambda^D. \end{gathered} \tag{6.56}$$

Note that we have to assume that the basis vectors A_a of the extended vector space $\mathcal{T}$ are not affected by the dilatations. We also have to assume a specific transformation property of the Higgs field Ξ. With the same assumptions we find that the Lagrangian (5.14) of the internal gauge fields is invariant and the Lagrangian (5.26) of the Higgs scalar field is also invariant if we disregard the potential term $-V(\Xi)\eta$.

In conclusion, if we put $m = 4$, the complete Lagrangian form of gravitation and elementary particles is invariant, apart from the terms containing the cosmological constant and the Higgs potential (one can redefine the Higgs potential in such a way that it includes the cosmological constant). If we take these terms into account, we have

$$\delta\lambda = -\delta(V(\Xi)\eta) - \delta(k_4\eta) = \zeta(-4\lambda v^2(\Xi^\dagger\Xi - v^2) + 4k_4)\eta, \tag{6.57}$$

where we have adopted the expression (5.61) for the Higgs potential and we have assumed that k_4 is constant. This simple result is a further indication that $m = 4$ is an interesting choice.

Note that the potential term that is necessary in order to obtain the spontaneous symmetry breaking of the internal gauge symmetry is also responsible for the explicit symmetry breaking of the total dilatation symmetry. If one likes to preserve the total dilatation symmetry, one has to invent a different way to generate a nonvanishing vacuum expectation value of the Higgs field Ξ.

For instance, it could have a cosmological origin, as it happens for the Brans-Dicke field Φ. However, a model of this kind would imply the existence of unobserved zero mass Higgs particles. Note that also the Brans-Dicke field should describe unobserved zero-mass particles. In any case, it is very difficult to modify the Standard Model of elementary particles without spoiling its very good agreement with the experimental observations. A serious discussion of these problems lies outside the scope of the present notes.

In analogy with eq. (4.41), one can define (using the real formalism) the 3-form

$$\theta_D = -\omega^i \wedge \sigma_i - 2^{-1}\omega^{[ik]} \wedge \sigma_{[ik]} + \Xi^T \pi^S. \tag{6.58}$$

Note that the internal gauge fields and the Dirac fields do not contribute. For $m = 4$ we have some relevant cancellations and we obtain (introducing the complex formalism for Ξ)

$$\theta_D = 6\alpha\Phi\chi^i\eta_i - 2^{-1}A^i(\Xi^\dagger\Xi)\eta_i. \tag{6.59}$$

This form is conserved (namely it is closed) only if $\delta\lambda = 0$. Otherwise we have

$$d\theta_D = (-4\lambda v^2(\Xi^\dagger\Xi - v^2) + 4k_4)\eta. \tag{6.60}$$

Note that the right hand side vanishes if $k_4 = 0$ and Ξ takes its vacuum expectation value.

By means of eq. (4.69) we obtain the more explicit formula

$$\begin{aligned} 6\alpha A_i(\Phi\chi^i) - 6\alpha\Phi\chi_i\chi^i - 2^{-1}A^iA_i(\Xi^\dagger\Xi) + 2^{-1}\chi^iA_i(\Xi^\dagger\Xi) \\ = -4\lambda v^2(\Xi^\dagger\Xi - v^2) + 4k_4. \end{aligned} \tag{6.61}$$

The same equation can also be obtained directly from the field equations, but the derivation given above helps us to understand their meaning and may suggest further developments.

6.4 Lagrangian constraints and pre-symplectic double forms

It is interesting to apply the pre-symplectic formalism described in Section 4.5 to the specific models defined in the preceding Sections. We consider first the Einstein-Cartan theory of Section 5.1. The normal equations (6.5) do not contain the "velocities" namely the structure coefficients, and therefore coincide with the primary constraints. By sustituting them into eq. (4.79), we obtain

$$\begin{aligned}\theta' &= k\epsilon_{ikjl}\,\hat{d}\omega^{[ik]}\wedge\omega^j\wedge\omega^l - k_1\epsilon_{ikjl}\hat{d}(\omega^{[ik]}\wedge\omega^j\wedge\omega^l)\\ &+3^{-1}k_2\epsilon_{ikjn}g_{lm}\hat{d}(\omega^{[ik]}\wedge\omega^{[jl]}\wedge\omega^{[mn]}) + 2k_5\chi^k\hat{d}\eta_k. \end{aligned}\tag{6.62}$$

Since we have $k_5 = 0$ and the other coefficients k, k_1, k_2 are constant, we obtain immediately the pre-symplectic double form

$$\Omega' = -\hat{d}\theta = k\epsilon_{ikjl}\,\hat{d}\omega^{[ik]}\wedge\hat{d}(\omega^j\wedge\omega^l) = 2k\epsilon_{ikjl}\,\hat{d}\omega^{[ik]}\wedge\hat{d}\omega^j\wedge\omega^l. \tag{6.63}$$

One must keep in mind that only the restriction of this form to the 3-dimensional surface Σ contributes to the pre-symplectic form $\Omega(\Sigma)$.

In the scalar-tensor theory of Sections 6.1 and 6.2 the coefficients are given by eqs. (6.25) and (6.30) and the normal equations contain the structure coefficients through the fields ϕ and χ_k. In order to obtain the constraint equations, we have to express them as functions of the "canonical momenta" σ_α, for instance (for $m \neq 1, 2$) by means of the equations

$$\epsilon^{mnik}i(A_m)i(A_n)\sigma_{ik} = 24(m-2)(m-1)^{-1}\phi^{m-1}, \tag{6.64}$$

$$\epsilon^{mnik}i(A_m)i(A_n)\sigma_i = 12\alpha\phi^{m-2}\chi^k, \tag{6.65}$$

that follow from eq. (6.5). Remember that the vector fields A_α are uniquely determined by the 1-forms ω^α.

In this way we obtain the pre-symplectic double form

$$\begin{aligned}\Omega' &= 2^{-2}\epsilon_{ikjl}\hat{d}(\phi^{m-1})\wedge\hat{d}\omega^{[ik]}\wedge\omega^j\wedge\omega^l + 2^{-2}\phi^{m-1}\epsilon_{ikjl}\,\hat{d}\omega^{[ik]}\wedge\hat{d}(\omega^j\wedge\omega^l)\\ &-2^{-2}(m-1)^{-1}\epsilon_{ikjl}\hat{d}(\phi^{m-1})\wedge\hat{d}(\omega^{[ik]}\wedge\omega^j\wedge\omega^l)\\ &-2\alpha\hat{d}(\phi^{m-2}\chi^k)\wedge\hat{d}\eta_k. \end{aligned}\tag{6.66}$$

If we introduce the new variables Φ and $\tilde{\omega}^{ik}$ by means of eqs. (6.32) and (6.38), after some calculations, we get

$$\Omega' = 2^{-2}\epsilon_{ikjl}\,\hat{d}\tilde{\omega}^{[ik]} \wedge \hat{d}(\Phi\omega^j \wedge \omega^l) - 2\alpha\hat{d}(\Phi\chi^k) \wedge \hat{d}\eta_k. \tag{6.67}$$

Through this change of variables (that depends on m) we have obtained a pre-symplectic form that, as the field equations, does not contain the parameter m. This means that m should also be absent in an hypothetic quantized theory, as soon as a parameter ℓ is not introduced.

If $n > 10$, namely if internal gauge fields are present, one has to take into account the primary constraint following from eq. (5.12), namely

$$i(A_{[ik]})\sigma_a = i(A_b)\sigma_a = 0. \tag{6.68}$$

The contribution of the internal gauge fields to the pre-symplectic double form is formally unchanged, namely

$$\Omega' = \hat{d}\omega^a \wedge \hat{d}\sigma_a, \tag{6.69}$$

but it is defined in the submanifold Γ'. It may be difficult to find the connection with the primary constraint of the usual formalism in the spacetime $\mathcal{M}$. It is based on a different choice of the unconstrained phase space Γ, but what is relevant is the physical phase space.

If there is a scalar field Ξ of the kind described in Section 5.3, from eq. (5.28) we obtain the primary constraints

$$i(A_{[ik]})\pi^S = i(A_b)\pi^S = 0 \tag{6.70}$$

and the contribution to the pre-symplectic double form maintains the form

$$\Omega' = \hat{d}\Xi^T \wedge \hat{d}\pi. \tag{6.71}$$

Finally, we have to consider the Dirac fields. Eq. (5.37) does not contain the "velocities" and therefore provides directly the primary constraint. The contribution to the pre-symplectic double form is

$$\Omega' = -i\hat{d}\Psi^T C\gamma^i \wedge \hat{d}\Psi\eta_i. \tag{6.72}$$

A new feature is that the fields Ψ are anticommuting and this expression does not vanish because the matrices $C\gamma^i$, according to eq. (3.15), are symmetric.

Chapter 7

Classical field theories with $Sp(4, \mathbf{R})$ symmetry (not complete)

7.1 Higher symmetries and a substitution rule

In Section 2.5 we have introduced the geometric symmetry group of the field equations $\mathcal{F}^G$ and in Section 3.3 we have suggested that it is a subgroup of the symmetry group $GL(4, \mathbf{R})$ of the cone $\mathcal{T}^+$. In the present Chapter we deal with field theories with a geometric symmetry group $\mathcal{F}^G$ larger than the Lorentz group. In this way we introduce the new fundamental constant ℓ.

A detailed analysis of the possible geometric symmetry groups containing the Lorentz group was given in ref. [6]. Since at that time the explicit forms of the cone $\mathcal{T}^+$ and of its symmetry group were not known, one had to consider all the sugroups of the group $GL(10, \mathbf{R})$ containing all the linear transformations of $\mathcal{T}$ and a definite choice could not be obtained. In this analysis the most relevant group $GL(4, \mathbf{R})$ was omitted and an erratum [6] was published to correct this mistake.

The higher symmetry theories should not contradict the normal Lorentz symmetric theories treated in Chapter 5 in their range of validity. Einstein's General Relativity has been confirmed with high precision by laboratory experiments, for instance on the equivalence principle, and by accurate observations of the planetary system and of the binary pulsars. For an updated review, see ref. [109]. The Einstein-Cartan theory is not, at present, experimentally distinguishable from General Relativity and we prefer it in the present notes only because it gives a more symmetric treatment of energy-momentum and spin, in agreement with the equity principle (see Section 2.4). Also the Brans-Dicke scalar-tensor theory cannot be distinguished from General Relativity if the parameter ω is sufficently large.

There is some possibility for modifications of Einstein's gravitational theory at very small distances, or very large curvatures, and at very large distances, which appear in the galactic and cosmological observations. It may look strange that the phenomena at very large distance can be influenced by the introduction of a very small fundamental length. It is possible, however, that very small local effects due to the fundamental length accumulate over very long distances giving observable effects. This idea has been discussed, in a different context, in ref. [111].

In the search of new theories, it is important to take into account a "correspondence principle" that requires that the old theory is, in some way, a limit of the new theory. The simplest example is the nonrelativistic limit of a relativistic particle. If we consider the particle Lagrangians as differential 1-forms and we reintroduce the symbol c for the velocity of light, we have

$$\lim_{c\to\infty}\left(-mc^2(1-c^{-2}\|\dot{\mathbf{x}}\|^2)^{1/2}dt + d(mc^2t)\right) = 2^{-1}m\|\dot{\mathbf{x}}\|^2dt, \tag{7.1}$$

namely the limit of the Lorentz invariant Lagrangian, after the subtraction of a divergent exact form, gives the nonrelativistic Lagrangian. The contraction of the Lorentz group is the Galilei group and under the Galilei transformation

$$\mathbf{x} \to \mathbf{x} + \mathbf{v}t \tag{7.2}$$

we have

$$2^{-1}m\|\dot{\mathbf{x}}\|^2dt \to 2^{-1}m\|\dot{\mathbf{x}}\|^2dt + d\left(m\mathbf{v}\cdot\mathbf{x} + 2^{-1}m\|\mathbf{v}\|^2t\right). \tag{7.3}$$

We see that the new Lagrangian is *quasi-invariant*, namely invariant up to an additional exact form that does not affect the equations of motion.

We could expect a similar situation when we consider the *normal limit* $\ell \to 0$ of a theory with higher symmetry, but the Lagrangian forms of the normal theories examined in Chapter 5 are not quasi-invariant with respect to any of the contracted transformations given by eqs. (3.81) and (3.82). This remark could be discouraging, but we have to consider that our problem has some more complications. In particular, as we have remarked in Section 3.7, the higher symmetry group is spontaneously broken and there are several nonsymmetric vacuum states. It follows that it is not sufficient to consider the limit $\ell \to 0$, but one has also to choose one of the degenerate vacuum states. The new Lagrangian should approach the old Lagrangian only for configurations which are, in some sense, near to one of the vacuum states, that becomes the unique vacuum state of the old theory. In this way the quasi-invariance of the old Lagrangian with respect to the contracted high-symmetry group is lost.

In order to obtain a gravitational Lagrangian form with higher symmetry by means of a minimal modification of a known Lorentz invariant Lagrangian form of the kind considered in Chapter 5, one can rewrite the latter by using the spinor formalism, namely introducing the forms $\omega^{(AB)}$ and the structure coefficients $F^{(EF)}_{(AB)(CD)}$. The expression obtained in this way must also contain the antisymmetric constant Lorentz invariant spinors C_{AB} and G_{AB} defined in Sections 3.3 and 3.5, that, however, are not invariant under $GL(4, \mathbf{R})$. A Lagrangian symmetric under a larger group can be obtained by replacing these constant spinors by antisymmetric spinor expressions obtained from the structure coefficients by means of the rules of the spinor calculus. We call this procedure the *substitution rule*.

Of course, the new expressions must be very near to the constant quantities C or G in the physical situations in which the old theory is valid, in particular in the Poincaré vacuum. As a matter of fact, it is impossible to find an expression that replaces G, because it is odd under space inversion (represented by γ_0) while the Poincaré structure constants are invariant under space inversion. Instead, a very natural replacement for C can be found in terms of the spinor t_{AB} defined in Section 3.7 by eq. (3.89). If we replace only C, we obtain a Lagrangian symmetric with respect to the subgroup of $GL(4, \mathbf{R})$ that leaves G invariant, namely the axial symplectic group $Sp(4, \mathbf{R})_A$ introduced in Section 3.6. It is possible that theories invariant with respect to $GL(4, \mathbf{R})$ or $SL(4, \mathbf{R})$, can be found in a more general geometric scheme, but some new idea is necessary.

Actually, in the construction of Lagrangians symmetric with respect to $Sp(4, \mathbf{R})_A$ it is convenient to use the 5-vector f_u defined by eq. (3.93) instead of the 6-vector t_u equivalent to the antisymmetric spinor t_{AB}, because we have $f_i = 0$ in all the theories cosidered in Chapter 5. With this choice, the substitution rule takes thye form

$$C \to \phi^{-1} f_u \breve{\Theta}^u = \phi^{-1} C(f_4 + f_i \gamma^i \gamma^5) = C(\psi_4 + \psi_i \gamma^i \gamma^5), \qquad (7.4)$$

where

$$\phi = (-f_u f^u)^{1/2}, \qquad \psi_u = \phi^{-1} f_u, \qquad \psi_u \psi^u = -1. \tag{7.5}$$

As a consequence, we also have

$$\gamma_5 = G^{-1} C \to (\psi_4 \gamma_5 - \psi_i \gamma^i). \tag{7.6}$$

This substitution tends to an equality in the limit

$$\psi \to \hat{\psi} = (0, 0, 0, 0, 1), \tag{7.7}$$

in particular when the geometry of $\mathcal{S}$ approaches the geometry of a principal bundle of frames, as we have shown in Section 3.7. We are assuming the inequality

$$(f_4)^2 > f_i f^i, \tag{7.8}$$

that is satisfied if the structure coefficients are not too different from the structure constants of the Poincaré group. We cannot assume $\psi_4 \geq 0$, because a transformation of $Sp(4, \mathbf{R})_A$ can change the sign of this variable. We shall often use an approximation in which the quantities ψ_i are infinitesimal and $\psi_4 - 1$ is infinitesimal of the second order.

The substitution rule can also be treated in a more abstract and general way, also working for the matter fields. We indicate by Z all the geometric objects and the other dynamical fields on which the Lagrangian λ depends and by $\Delta(a)$ the representation of $Sp(4, \mathbf{R})_A$ acting on them. A normal Lagrangian form $\lambda(Z)$ of the kind treated in Chapter 5 has the Lorentz invariance property

$$\lambda(\Delta(a) Z) = \lambda(Z), \qquad a \in SL(2, \mathbf{C}). \tag{7.9}$$

After the substitution rule, we obtain the new Lagrangian form $\lambda(Z, \psi)$ that also contains the normalized 5-vector ψ and is defined by

$$\lambda(Z, \psi) = \lambda(\Delta(a) Z), \tag{7.10}$$

where a is an element with the properties

$$a \in Sp(4, \mathbf{R})_A, \qquad \Delta(a)\psi = \hat{\psi}. \tag{7.11}$$

Since the 5-vector $\hat{\psi}$ is Lorentz invariant, this condition determines the element a up to the multiplication on the left by an element of $SL(2, \mathbf{C})$, but this ambiguity does not influence the result because λ satisfies the invariance relation (7.9). We have $\lambda(Z, \hat{\psi}) = \lambda(Z)$ and

$$\lambda(\Delta(a)Z, \Delta(a)\psi) = \lambda(Z, \psi), \qquad a \in Sp(4, \mathbf{R})_A, \tag{7.12}$$

namely the new Lagrangian is invariant under $Sp(4, \mathbf{R})_A$. These two properties characterize the result of the substitution rule.

We can also apply the substitution rule to other Lorentz invariant expressions, to be used as building blocks in the construction of invariant Lagrangians. In particular starting from the 4-form η defined in Section 0.3 one obtains

$$\eta(\psi) = -(24)^{-1}\ell^4 \psi_{u'}\psi_{v'}\psi_{w'}\psi_{x'}\psi^y \epsilon_{uvwxy}\omega^{[uu'5]}\wedge\omega^{[vv'5]}\wedge\omega^{[ww'5]}\wedge\omega^{[xx'5]}. \tag{7.13}$$

One can immediately see that this expression is invariant under $Sp(4, \mathbf{R})_A$ and that $\eta(\hat{\psi}) = \eta$.

We shall also use the 3-forms

$$\eta_\alpha(\psi) = i(A_\alpha)\eta(\psi) \tag{7.14}$$

with the properties

$$\eta_i(\hat{\psi}) = \eta_i, \qquad \eta_{[ik]}(\hat{\psi}) = 0. \tag{7.15}$$

Another useful formula is

$$\left(\frac{\partial \eta(\psi)}{\partial \psi_i}\right)_{\psi=\hat{\psi}} = -\ell\omega^{[ik5]} \wedge \eta_k = 2^{-1}\ell\epsilon^{ik}{}_{jl}\omega^{[jl]} \wedge \eta_k. \tag{7.16}$$

7.2 Normal field equations and use of the symmetry property.

The Lagrangians studied in Chapter 5 have a particular structure that permits a simpler treatment of the normal field equations. In the present Chapter too, we consider Lagrangian of the form

$$\lambda = d\omega^\alpha \wedge \sigma_\alpha + \lambda^A + \lambda^M, \tag{7.17}$$

where the quantities σ_α, λ^A, and λ^M contain, besides the forms ω^α, the field ϕ defined by eq. (7.5) and the fields

$$\chi_\alpha = F^\beta_{\alpha\beta}, \qquad \chi_i = F^j_{ij} + 2^{-1}F^{[jl]}_{i[jl]}, \qquad \chi_{[ik]} = F^j_{[ik]j} + 2^{-1}F^{[jl]}_{[ik][jl]}. \tag{7.18}$$

Note that these fields are not exactly equal to the fields indicated by the same symbols in eqs. (6.1) and (6.2). They have similar properties, but transform in a simpler way. After the application of the substitution rule, also the fields ψ_u appear in the Lagrangian. Instead of the fields ϕ and ψ_u, one can use directly the fields f_u.

The derivatives of λ with respect to the structure coefficients that appear in the normal field equation (4.28) contain two contributions, one coming from the exterior derivatives $d\omega^\alpha$ and the other originated by the dependence of λ on the quantities f_u and χ_α. The first contribution satisfies the normal field equations automatically and the second contribution gives the condition

$$\left(\frac{\partial f_u}{\partial F^\eta_{\epsilon\zeta}}\omega^\theta + \frac{\partial f_u}{\partial F^\eta_{\theta\zeta}}\omega^\epsilon\right) \wedge \frac{\partial \lambda}{\partial f_u} + \left(\frac{\partial \chi_\alpha}{\partial F^\eta_{\epsilon\zeta}}\omega^\theta + \frac{\partial \chi_\alpha}{\partial F^\eta_{\theta\zeta}}\omega^\epsilon\right) \wedge \frac{\partial \lambda}{\partial \chi_\alpha} = 0, \tag{7.19}$$

or, more explicitly, by using the 5-dimensional tensor notation and eqs. (3.93) and (7.18),

$$\begin{aligned}\alpha\Big(&(\delta^v_u \epsilon_{yy'}{}^{v'ww'} - \delta^{v'}_u \epsilon_{yy'}{}^{vww'} - \delta^w_u \epsilon_{yy'}{}^{w'vv'} + \delta^{w'}_u \epsilon_{yy'}{}^{wvv'})\omega^{[xx'5]} \\ &+(\delta^x_u \epsilon_{yy'}{}^{x'ww'} - \delta^{x'}_u \epsilon_{yy'}{}^{xww'} - \delta^w_u \epsilon_{yy'}{}^{w'xx'} + \delta^{w'}_u \epsilon_{yy'}{}^{wxx'})\omega^{[vv'5]}\Big) \wedge \tfrac{\partial\lambda}{\partial f_u} \\ &+\beta\Big((\delta^{vv'}_{uu'}\delta^{ww'}_{yy'} - \delta^{ww'}_{uu'}\delta^{vv'}_{yy'})\omega^{[xx'5]} \\ &+(\delta^{xx'}_{uu'}\delta^{ww'}_{yy'} - \delta^{ww'}_{uu'}\delta^{xx'}_{yy'})\omega^{[vv'5]}\Big) \wedge \tfrac{\partial\lambda}{\partial\chi_{[uu'5]}} = 0,\end{aligned} \tag{7.20}$$

where α and β are irrelevant constant coefficients and

$$\delta^{vv'}_{uu'} = \delta^v_u \delta^{v'}_{u'} - \delta^v_{u'}\delta^{v'}_u . \tag{7.21}$$

We fix arbitrarily the indices $u = v = y$ and $x \neq x'$, and without loss of generality we assume, for instance, that $u \neq x$. Then we chose $y' \neq y, x, x'$ and $v' \neq v, x, x', y'$. We choose the other indices w, w' in such a way that y, y', v', w, w' are all different. Then only the first term of eq. (7.20) survives and we have

$$\omega^{[xx'5]} \wedge \frac{\partial\lambda}{\partial f_u} = 0. \tag{7.22}$$

By reasoning as in the case of eq. (6.8) we obtain

$$\frac{\partial\lambda}{\partial f_u} = 0. \tag{7.23}$$

In order to treat the second part of eq. (7.19), we write it in the form

$$\Big((\delta^\epsilon_\alpha \delta^\zeta_\eta - \delta^\zeta_\alpha \delta^\epsilon_\eta)\omega^\theta + (\delta^\theta_\alpha \delta^\zeta_\eta - \delta^\zeta_\alpha \delta^\theta_\eta)\omega^\epsilon\Big) \wedge \frac{\partial \lambda}{\partial \chi_\alpha} = 0. \tag{7.24}$$

We choose arbitrarily the indices $\alpha = \zeta$ and θ. It is always possible to put $\epsilon = \eta \neq \alpha, \theta$ and we obtain

$$\omega^\theta \wedge \frac{\partial \lambda}{\partial \chi_\alpha} = 0. \tag{7.25}$$

It follows that

$$\frac{\partial \lambda}{\partial \chi_\alpha} = 0. \tag{7.26}$$

For the particular kind of Lagrangians we are considering, eqs. (7.23) and (7.26) are equivalent to the normal equation (4.28). By introducing the variables ϕ, ψ_i, eq (7.23) can also be written in the form

$$\frac{\partial \lambda}{\partial \psi_i} = 0 \tag{7.27}$$

$$\frac{\partial \lambda}{\partial \phi} = 0. \tag{7.28}$$

From eq. (7.17) and the normal field equations (7.23) and (7.26) we have

$$\frac{\partial \lambda}{\partial F^\gamma_{\alpha\beta}} = \left(\frac{\partial \lambda}{\partial F^\gamma_{\alpha\beta}}\right)_E = -2^{-1}\omega^\alpha \wedge \omega^\beta \wedge \sigma_\gamma \tag{7.29}$$

in agreement with eq. (4.31). The subscript E means that the partial derivative takes into account only the explicit dependence of λ on the structure coefficients and not the indirect dependence through the quantities f_u and χ_α.

The conservation laws corresponding to the infinitesimal $Sp(4, \mathbf{R})_A$ symmetry transformations play an important role in the following discussion. As it is explained in Section 4.5, these transformations are generated by the vector fields $X_{[uv]}$ (in the phase space) that act on the dynamical variables in the following way

$$\begin{gathered} X_{[uv]}\omega^{[xy5]} = \delta^x_u g_{vz}\omega^{[zy5]} + \delta^y_u g_{vz}\omega^{[xz5]} - \delta^x_v g_{uz}\omega^{[zy5]} - \delta^y_v g_{uz}\omega^{[xz5]}, \\ X_{[uv]}\phi = 0, \qquad X_{[uv]}\psi_w = g_{uw}\psi_v - g_{vw}\psi_u, \\ X_{[uv]}\chi_{[xy5]} = g_{xu}\delta^z_v\chi_{[zy5]} + g_{yu}\delta^z_v\chi_{[xz5]} - g_{xv}\delta^z_u\chi_{[zy5]} - g_{yv}\delta^z_u\chi_{[xz5]}. \end{gathered} \tag{7.30}$$

According to eq. (4.41), this symmetry gives rise to the conservation laws

$$d\theta_{[uv]} = 0, \qquad \theta_{[uv]} = \theta^G_{[uv]} + \theta^M_{[uv]}, \tag{7.31}$$

where

$$\theta^G_{[uv]} = X_{[uv]}\omega^\alpha \wedge \sigma_\alpha, \qquad \theta^M_{[uv]} = X_{[uv]}\Psi^U \pi_U. \tag{7.32}$$

These conservation laws, are a consequence of all the field equations and we want to show that, when all the other field equations are satisfied, the normal equation (7.27) is equivalent to the following equation that sometimes has an easier treatment and a more direct meaning:

$$\psi^u d\theta_{[uv]} = 0. \tag{7.33}$$

We use the invariance property of λ, which, using the shorthand notation introduced in Section 7.1, can be written in the form

$$X_{[uv]}\lambda = X_{[uv]}Z\frac{\partial\lambda}{\partial Z} + X_{[uv]}\psi_w\frac{\partial\lambda}{\partial\psi_w} = 0, \tag{7.34}$$

or, more explicitly,

$$\begin{gathered} X_{[uv]}\Psi^U\tfrac{\partial\lambda}{\partial\Psi^U} + X_{[uv]}(A_\alpha\Psi^U)\tfrac{\partial\lambda}{\partial A_\alpha\Psi^U} \\ -2^{-1}X_{[uv]}F^\gamma_{\alpha\beta}\omega^\alpha\wedge\omega^\beta\wedge\sigma_\gamma + X_{[uv]}\omega^\alpha i_\alpha\lambda + X_{[uv]}\psi_w\tfrac{\partial\lambda}{\partial\psi_w} = 0, \end{gathered} \tag{7.35}$$

where eqs. (7.26) and (7.29) have been taken into account.

By means of the normal field equation (4.29) and of the tangential equations (4.34), (4.35) and (4.36), we obtain

$$\begin{gathered} d(X_{[uv]}\Psi^U\pi_U) - A_\alpha\Psi^U(X_{[uv]}\omega^\alpha)\wedge\pi_U + d(X_{[uv]}\omega^\alpha)\wedge\sigma_\alpha \\ +F^\gamma_{\alpha\beta}X_{[uv]}\omega^\alpha\wedge\omega^\beta\wedge\sigma_\gamma + X_{[uv]}\omega^\alpha i_\alpha\lambda + X_{[uv]}\psi_w\tfrac{\partial\lambda}{\partial\psi_w} \\ = d(X_{[uv]}\Psi^U\pi_U) - (X_{[uv]}\omega^\alpha)\wedge\tau_\alpha \\ +d(X_{[uv]}\omega^\alpha)\wedge\sigma_\alpha + X_{[uv]}\psi_w\tfrac{\partial\lambda}{\partial\psi_w} = 0 \end{gathered} \tag{7.36}$$

and finally

$$d\theta_{[uv]} + (g_{uw}\psi_v - g_{vw}\psi_u)\frac{\partial\lambda}{\partial\psi_w} = 0. \tag{7.37}$$

If the normal equation (7.27) is satisfied (as well as all the other field equations) this is just a new proof of the conservation of $\theta_{[uv]}$. Conversely, if we assume eq. (7.33), we easily obtain the normal equation (7.27), as we have anticipated above.

In the analysis of the field equations, a considerable simplification can be obtained by choosing an *adapted basis* in the space $\mathcal{T}$ by means of a suitable (global) transformation of the symmetry group $Sp(4, \mathbf{R})_A$. In this way, assuming the inequality (7.8), we can put, at a single *distinguished point* $\hat{s} \in \mathcal{S}$, $f_i = 0$ or $\psi_i = 0$. At this point the Lagrangian form λ is not affected by the substitution rule. Besides λ, also π, σ_α, $\theta_{[uv]}$, and τ_α have, at the particular point $\hat{s}$, the same form they have before the application of the substitution rule. In particular, the quantities τ^M_α have the local form (4.68).

However, one has to be careful in dealing with expressions containing derivatives with respect to ψ_i or other differential operators applied to expressions containing ψ_i: one has to compute the derivatives first and then to put $\psi_i = 0$. For instance, the field equations and the conservation laws may contain new terms that do not vanish at the point $\hat{s}$. In order to compute them, we need the derivatives of various quantities with respect to ψ_i at the point $\hat{s}$.

In some simple cases, these derivatives can be obtained directly or by means of eq. (7.16). They can also be obtained by means of the covariance of these quantities with respect to the infinitesimal transformations $X[j4]$ given by

$$\begin{gathered} X_{[i4]}\Psi = \Sigma_{[i4]}\Psi, \\ X_{[i4]}\omega^j = -2^{-1}\ell\epsilon^j{}_{ikl}\omega^{[kl]}, \qquad X_{[i4]}\omega^{[jk]} = -\ell^{-1}\epsilon^{jk}{}_{il}\omega^l, \\ X_{[i4]}\chi_j = -2^{-1}\ell^{-1}\epsilon_{ji}{}^{kl}\chi_{[kl]} \qquad X_{[i4]}\chi_{[jk]} = -\ell\epsilon_{jki}{}^l\chi_l. \end{gathered} \tag{7.38}$$

$$\begin{gathered} X_{[i4]}\pi = -\Sigma^T_{[i4]}\pi, \\ X_{[i4]}\sigma_j = -2^{-1}\ell^{-1}\epsilon_{ji}{}^{kl}\sigma_{[kl]}, \qquad X_{[i4]}\sigma_{[jk]} = -\ell\epsilon_{jki}{}^l\sigma_l, \\ X_{[i4]}\theta_{[jk]} = g_{ik}\theta_{[j4]} - g_{ij}\theta_{[k4]}, \qquad X_{[i4]}\theta_{[j4]} = \theta_{[ji]}. \end{gathered} \tag{7.39}$$

By using the shorthand notation introduced at the end of Section 7.1, since at the point $\hat{s}$ it is

$$X_{[i4]}\psi_k = g_{ik}\psi_4 = g_{ik}, \qquad X_{[i4]}\psi_4 = \psi_i = 0, \tag{7.40}$$

we obtain, at the same point,

$$\tfrac{\partial\pi}{\partial\psi^i} = -\Sigma^T_{[i4]}\pi - \tfrac{\partial\pi}{\partial Z}X_{[i4]}Z,$$

$$
\begin{aligned}
\frac{\partial\sigma_j}{\partial\psi^i} &= -2^{-1}\ell^{-1}\epsilon_{ji}{}^{kl}\sigma_{[kl]} - \frac{\partial\sigma_j}{\partial Z}X_{[i4]}Z,\\
\frac{\partial\sigma_{[jk]}}{\partial\psi^i} &= -\ell\epsilon_{jki}{}^{l}\sigma_l - \frac{\partial\sigma_{[jk]}}{\partial Z}X_{[i4]}Z,\\
\frac{\partial\theta_{[jk]}}{\partial\psi^i} &= g_{ik}\theta_{[j4]} - g_{ij}\theta_{[k4]} - \frac{\partial\theta_{[jk]}}{\partial Z}X_{[i4]}Z,\\
\frac{\partial\theta_{[j4]}}{\partial\psi^i} &= \theta_{[ji]} - \frac{\partial\theta_{[j4]}}{\partial Z}X_{[i4]}Z.
\end{aligned}
\tag{7.41}
$$

In many situations the "old" theory, considered before the application of the substitution rule, agrees with a good accuracy with the empirical data. As a consequence, it is important to understand the conditions under which the solutions of the "old" theory are also solutions of the "new" theory obtained by means of the substitution rule. For these solutions we have $\psi = \hat{\psi}$ and the corrections to the field equations, proportional to derivatives of ψ vanish. However, the "new" theory has the additional normal equation (7.27), equivalent to the conservation law (7.33), that in the case we are considering takes the form $d\theta_{[i4]} = 0$. If and only if it astisfies this equation a solution of the "old" theory is also a solution of the "new" theory.

7.3 Two examples of Lagrangians invariant under $Sp(4, \mathbf{R})_A$.

In ref. [6] two Lagrangians with $Sp(4, \mathbf{R})_A$ symmetry have been suggested, showing that the corresponding theories have the Poincaré vacuum solution, besides the degenerate vacuum solutions obtained from it by the action of the symmetry group. The existence of nonvacuum solutions was not investigated, due to mathematical difficulties, and in the present Chapter we continue this study using the concepts developed in the preceding Sections. We find that these models are not completely satisfactory, but they provide a basis for the construction of more acceptable theories.

We use the conventions we have adopted in the present notes and the 5-dimensional tensor calculus for $SO(2,3)_V$ described in Section 3.6. In particular we use the 5-vectors f_u, ψ_u defined by eqs. (3.93), (7.5) and the notation $\omega^{[uv5]}$ for the 1-forms ω^α, in analogy with eq. (3.69). The index 5 is added for compatibility with the 6-dimensional formalism and to recall that the other two indices take the values $u, v = 0, \ldots, 4$ and transform according to $SO(2,3)_A$.

The first Lagrangian is

$$\lambda^H = -2\ell^2 k\psi_x\psi_y g_{uu'}g_{vv'}d\omega^{[uv5]}\wedge\omega^{[xu'5]}\wedge\omega^{[yv'5]},$$
$$\lambda^A = \ell^2 k\psi_x\psi_y\psi^z g_{uu'}\epsilon_{vv'ww'z}\omega^{[ww'5]}\wedge\omega^{[uv5]}\wedge\omega^{[xu'5]}\wedge\omega^{[yv'5]}. \quad (7.42)$$

It contains the fields ψ_u, but not the field ϕ and the normal equation (7.28) is trivially satisfied. We have seen in Section 7.2 that the normal equation (7.27) is equivalent to the conservation of $\theta_{[uv]}$ and we do not need to consider it. If the normal field equations are satisfied, we have

$$\sigma_{[uv5]} = -4\ell^2 k\psi_x\psi_y g_{uu'}g_{vv'}\omega^{[xu'5]}\wedge\omega^{[yv'5]}. \quad (7.43)$$

In order to find the primary constraints, we have to express the quantity $\psi_x\psi_y$, that contains the structure coefficients, in terms of the "canonical momenta." From eq. (7.43) we have

$$g^{uu'}g^{vv'}i(A_{[xu'5]})i(A_{[yv'5]})\sigma_{[uv5]} = 24\ell^2 k\psi_x\psi_y \quad (7.44)$$

and the pre-symplectic double form can be written as

$$\Omega' = -\hat{d}\theta' = -2\ell^2 k g_{uu'}g_{vv'}\hat{d}\omega^{[uv5]}\wedge\hat{d}(\psi_x\psi_y\omega^{[xu'5]}\wedge\omega^{[yv'5]}), \quad (7.45)$$

$$\theta' = -2\ell^2 k\psi_x\psi_y g_{uu'}g_{vv'}\hat{d}\omega^{[uv5]}\wedge\omega^{[xu'5]}\wedge\omega^{[yv'5]}. \quad (7.46)$$

One can see that for $\psi = \hat{\psi}$ the Lagrangian (7.42) coincides with the Regge-Ne'eman Lagrangian given in eqs. (5.1) and (5.2) with $k_1 = k_2 = k_4 = 0$ and $k_3 = k$ and the pre-symplectic form (7.45) coincides with eq. (6.63). It follows that the model we are considering can be obtained from the Regge-Ne'eman Lagrangian by means of the substitution rule.

The conserved 3-forms (7.32) are

$$\theta^G_{[uv]} = \hat{i}(X_{[uv]})\theta' = g_{vw}\omega^{[ww'5]}\wedge\sigma_{[uw'5]} - g_{uw}\omega^{[ww'5]}\wedge\sigma_{[vw'5]}$$
$$= -4\ell^2 k\psi_x\psi_y(g_{uu'}g_{vv'} - g_{uv'}g_{vu'})g_{ww'}\omega^{[v'w5]}\wedge\omega^{[xu'5]}\wedge\omega^{[yw'5]}. \quad (7.47)$$

An important consequence is

$$\psi^u\theta^G_{[uv]} = 0 \quad (7.48)$$

showing that a solution of the Einstein-Cartan theory is also a solution of the theory we are considering if and only if $d\theta^M_{[4k]} = 0$. From eqs. (4.69) and (5.49) we obtain

$$\theta^M_{[4k]} = W^5\eta_k, \qquad d\theta^M_{[4k]} = (A_kW^5 - F^j_{kj}W^5)\eta + \phi W^5 g_{kn}\omega^{[mn]}\wedge\eta_m. \quad (7.49)$$

In the model we are considering we have $F^j_{kj} = 0$ and $\phi = 1$, but these terms are necessary in the second model. We see that the condition is simply $W^5 = 0$.

From the conservation law (7.32) we obtain

$$d\psi^u \wedge \theta^G_{[uv]} = \psi^u d\theta^M_{[uv]}. \tag{7.50}$$

and we see that if the 5-vector ψ^u is constant, the right hand side must vanish. In order to discuss this equation with more detail, we choose an adapted basis in $T_{\hat{s}}\mathcal{S}$, so that at the particular point $\hat{s}$ we have $\psi = \hat{\psi}$ and therefore, in the 4-dimensional Lorentz formalism,

$$\theta^G_{[4i]} = 0, \qquad \theta^G_{[ik]} = 2k(g_{ij}\epsilon_{klmn} - g_{kj}\epsilon_{ilmn})\omega^{[mn]} \wedge \omega^j \wedge \omega^l, \tag{7.51}$$

$$\begin{aligned} d\psi^i \wedge \theta^G_{[ik]} = {} & 4k(g_{im}\delta^p_k\delta^q_n + g_{in}\delta^q_k\delta^p_m - g_{km}\delta^p_i\delta^q_n - g_{kn}\delta^q_i\delta^p_m)A_p\psi^i\omega^{[mn]} \wedge \eta_q \\ & + k(g_{ij}\epsilon_{klmn} - g_{kj}\epsilon_{ilmn})A_{[pq]}\psi^i\omega^{[pq]} \wedge \omega^{[mn]} \wedge \omega^j \wedge \omega^l. \end{aligned} \tag{7.52}$$

In the absence of Fermion fields, this expression must vanish and, after some calculations, we find that the derivatives of ψ_i must vanish too. If this happens in a connected region of $\mathcal{S}$, in this region we have $\psi_i = 0$ and all the equations coincide with the ones examined in Chapter 5. It is interesting to consider a situation in which the Fermion fields are very small and therefore ψ_i is very small too and eq. (7.49) has an approximate validity. A comparison of the preceding equations gives the approximate results, valid in an adapted frame,

$$A_k W^5 = 0, \qquad A_{[ik]}\psi_j = 0, \qquad A_i\psi_k = (2k)^{-1}W^5 g_{ik}. \tag{7.53}$$

From this equation we learn that the quantity W^5, related to the Fermion spin, is responsible for the appearance of nonvanishing (or nonconstant) values of ψ_i. If in a connected region $W^5 = 0$, the theory is perfectly equivalent to the Einstein-Cartan theory. We shall see in Chapter 9 that W^5 is extremely small if one excludes the interior of stars and the first few minutes of the big bang. It follows that the model presented in the present Section is in agreement with the observations.

Unfortunately, eq. (7.53) shows that W^5 must be a constant and this in an unacceptable condition on the Fermion fields. We conclude that the model is (at least) incomplete and that other gravitatinal degrees of freedom must be introduced in order to obtain a correct matching of the two sides of eq. (7.50). The first candidates are the fields χ_α defined in eq. (7.18), as we discuss in the following Section.

The second Lagrangian proposed in ref. [6], symmetric under total dilatations as in eq. (6.55), is

$$\lambda^H = -2^{-1}\ell^2\phi^{m-1}(\psi_x\psi_y + (m-1)^{-1}g_{xy})g_{uu'}g_{vv'}d\omega^{[uv5]} \wedge \omega^{[xu'5]} \wedge \omega^{[yv'5]},$$
$$\lambda^A = 2^{-2}\ell^2\phi^m\psi_x\psi_y\psi^z g_{uu'}\epsilon_{vv'ww'z}\omega^{[ww'5]} \wedge \omega^{[uv5]} \wedge \omega^{[xu'5]} \wedge \omega^{[yv'5]}. \quad (7.54)$$

One can see that for $\psi = \hat{\psi}$ it coincides with the Lagrangian studied in Sections 6.1 and 6.2 with

$$k = 2^{-2}\phi^{m-1}, \qquad k_1 = 2^{-2}(m-1)^{-1}\phi^{m-1},$$
$$k_2 = -\ell^2 k_1, \qquad k_3 = 2^{-2}\phi^m, \qquad k_4 = k_5 = 0. \quad (7.55)$$

Note that the field ϕ is defined in a different way. If we also take the limit $\ell \to 0$, we obtain the model of Section 6.2 without the cosmological term and the Lagrangian λ^χ proportional to k_5. We have shown in Section 6.1 that a model with $k_5 = 0$ has problems if massive particles are present and is, in any case, in disagreement with observations that suggest a very large value of the parameter ω.

In analogy with the first model, we have

$$\sigma_{[uv5]} = -\ell^2\phi^{m-1}(\psi_x\psi_y + (m-1)^{-1}g_{xy})g_{uu'}g_{vv'}\omega^{[xu'5]} \wedge \omega^{[yv'5]}, \quad (7.56)$$

$$\theta^G_{[uv]} = -\ell^2\phi^{m-1}\psi_x\psi_y(g_{uu'}g_{vv'} - g_{uv'}g_{vu'})g_{ww'}\omega^{[v'w5]} \wedge \omega^{[xu'5]} \wedge \omega^{[yw'5]} \quad (7.57)$$

and eqs. (7.48) and (7.50) are valis also in this case.

It is interesting to examine, as in the first model, the approximate explicit form of eq. (7.50) for small values of W^5 and of ψ_i and we obtain

$$A_k W^5 = F^j_{kj}W^5, \qquad A_{[ik]}\psi_j = 0, \qquad A_i\psi_k = 2\phi^{1-m}W^5 g_{ik}. \quad (7.58)$$

In this approximation the scalar-tensor theory of Sections 6.1 and 6.2 is approximately valid an we can use, eq. (6.41) to obtain the relation

$$A_k \ln W^5 = F^j_{kj} = -\omega A_k \ln\Phi, \qquad \Phi = C(W^5)^{-1/\omega}. \quad (7.59)$$

where C is a constant. This means that in an empty region, where $W^5 \to 0$, we have $\Phi \to \infty$ or $\Phi \to 0$, according to the sign of ω. This unacceptable feature can be avoided only if $1/\omega \to 0$ (as it is suggested by the astronomical observations) namely if $\alpha = 1/3$. In the model we are considering, without the additional term (6.3), we have $\alpha = 0$ and $\omega = -3/2$. This argument suggests that we have to add to the Lagrangian the additional term (6.3) with the appropriate coefficient.

Chapter 8

Test particles in geometric fields (not ready)

See ref. [8, 10, 21, 22, 24].

Chapter 9

Cosmological applications (not ready)

Chapter 10

Graded field algebras and antiderivations (not ready)

See ref. [15].

Chapter 11

Quantum fields in a fixed geometric background (not ready)

See ref. [12, 16, 17].

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